Theoretical Model for the Kramers-Moyal Description of Turbulence Cascades

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We derive the Kramers-Moyal equation for the conditional probability density of velocity increments from the theoretical model recently proposed by V. Yakhot [Phys. Rev. E **57**, 1737 (1998)] in the limit of the high Reynolds number. We show that the higher order ($n \geq 3$) Kramers-Moyal coefficients tend to zero and the velocity increments are evolved by the Fokker-Planck operator. Our results are compatible with the phenomenological description, developed for explaining recent experiments by R. Friedrich and J. Peinke [Phys. Rev. Lett. **78**, 863 (1997)]. [S0031-9007(99)08500-2]

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The problem of scaling behavior of longitudinal velocity difference $U = u(x_1) - u(x_2)$ in turbulence and the probability density function of U, i.e., $P(U)$, attracts a great deal of attention [1–7]. Statistical theory of turbulence has been brought forward by Kolmogorov [8] and further developed by others [9–12]. The approach is to model turbulence using stochastic partial differential equations. Kolmogorov conjectured that the scaling exponents are universal, independent of the statistics of large-scale fluctuations and the mechanism of the viscous damping, when the Reynolds number is sufficiently large. However, recently it has been found that there is a relation between the probability distribution function (PDF) of velocity and those of the external force (see [13] for more details). In this direction, Polyakov [1] has recently offered a field theoretic method to derive the probability distribution or density of states in $(1 + 1)$ dimensions in the problem of the randomly driven Burgers equation [14,15]. In one dimension, turbulence without pressure is described by the Burgers equation [see also [16] concerning the relation between the Burgers equation and the Kardar-Parisi-Zhang (KPZ) equation]. In the limit of the high Reynolds number, using the operator product expansion (OPE), Polyakov reduces the problem of computation of correlation functions in the inertial subrange, to the solution of a certain partial differential equation [17,18]. Yakhot recently [13,19] generalized the Polyakov approach in three dimensions and found a closed differential equation for the two-point generating function of the "longitudinal" velocity difference in the strong turbulence (see also [20] about the closed equation for the PDF of the velocity difference for two and three-dimensional turbulence without pressure). On the other hand, recently [21,22] from a detailed analysis of experimental data of a turbulent free jet, Friedrich and Pienke have been able to obtain a phenomenological description of the statistical properties of a turbulent cascade using a Fokker-Planck equation. In other words, they have seen that the conditional probability density of velocity increments satisfies the Chapman-Kolmogorov equation. Mathematically this is a necessary condition for the velocity increments to be a Markovian process in terms of length scales. By fitting the observational data they have succeeded in finding the different Kramers-Moyal (KM) coefficients, and they find that the approximations of the third and fourth order coefficients tend to zero, whereas the first and second coefficients have well-defined limits. Then by addressing the implications dictated by [23] theorem they have gotten a Fokker-Planck evolution operator. As an evolution equation for the probability density function of velocity increments, the Fokker-Planck equation has been used to give information on the changing shape of the distribution as a function of the length scale. By using this strategy the information on the observed intermittency of the turbulent cascade is verified. In their description and based on simplified assumptions on the drift and diffusion coefficients, they have considered two possible scenarios in order to indicate that both the Kolmogorov 41 and 62 scalings are recovered as possible behaviors in their phenomenological theory.

In this paper we derive the Kramers-Moyal equation from the Navier-Stokes equation and show how the higher order ($n \geq 3$) Kramers-Moyal coefficients tend to zero in the high Reynolds number limit. Therefore, we find the Fokker-Planck equation from first principles. We show that the breakdown of the Galilean invariance is responsible for the scale dependence of the Kramers-Moyal coefficients. Finally, using the path-integral expression for the PDF we show how small-scale statistics is affected by PDF's in the large scale and this is confirmed by Landau's remark that the large-scale fluctuations of turbulence production in the integral range can invalidate the Kolmogorov theory [9,10].

Our starting point is the Navier-Stokes equations,

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nu \nabla^2 \mathbf{v} - \frac{\nabla p}{\rho} + \mathbf{f}(\mathbf{x}, t), \qquad \nabla \cdot \mathbf{v} = 0 \tag{1}$$

for the Eulerian velocity $\mathbf{v}(\mathbf{x}, t)$ and the pressure p with viscosity ν, in N dimensions. The force $\mathbf{f}(\mathbf{x}, t)$ is the

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external stirring force, which injects energy into the system on a length scale L. More specifically, one can take, for instance, a Gaussian distributed random force, which is identified by its two moments,

$$\langle f_\mu(\mathbf{x},t) f_\nu(\mathbf{x}',t')\rangle = k(0)\delta(t-t')k_{\mu\nu}(\mathbf{x}-\mathbf{x}'), \quad (2)$$

and $\langle f_\mu(\mathbf{x},t)\rangle = 0$, where $\mu,\nu = x_1, x_2, \ldots, x_N$. The correlation function $k_{\mu\nu}(r)$ is normalized to unity at the origin and decays rapidly enough where r becomes larger or equal to the integral scale L.

The force-free NS equation is invariant under space-time translation, parity, and scaling transformation. Also it is invariant under the Galilean transformation, $x \to x + Vt$ and $v \to v + V$, where V is the constant velocity of the moving frame. Both boundary conditions and forcing can violate some or all of the symmetries of the force-free NS equation. However, it is usually assumed that in the high Reynolds number flow all symmetries of the NS equation are restored in the limit $r \to 0$ and $r \gg \eta$, where η is the dissipation scale where the viscous effects become important. This means that in this limit the root-mean square velocity fluctuations $u_{\rm rms} = \sqrt{\langle v^2\rangle}$, which are not invariant under the constant shift V, cannot enter the relations describing moments of velocity difference. Therefore, the effective equations for the inertial-range velocity correlation functions must have the symmetries of the original NS equation. For many years this assumption was the basis of turbulence theories. But based on the recent understanding of turbulence, some of the constraints on the allowed turbulence theories can be relaxed [13]. Polyakov's theory of the large-scale random force driven Burgers turbulence [1] was based on the assumption that weak small-scale velocity difference fluctuations (i.e., $|v(x+r) - v(x)| \ll u_{\rm rms}$ and $r \ll L$), where L is the integral scale of the system, obey the G-invariant dynamic equation, meaning that the integral scale and the single-point $u_{\rm rms}$ induced by random forcing cannot enter the resulting expression for the probability density. According to [13] it has been shown how the $u_{\rm rms}$ enters the equation for the PDF and therefore breaks the G invariance in the limited Polyakov's sense. We are interested in the scaling of the longitudinal structure function $S_q = \langle [u(x+r) - u(x)]^q\rangle = \langle U^q\rangle$, where $u(x)$ is the x component of the three-dimensional velocity field, and r is the displacement in the direction of the x axis. Let us define the generating function $\hat{Z}$ for the longitudinal structure function $\hat{Z} = \langle e^{\lambda U}\rangle$. According to [13] in the spherical coordinates the advective term in Eq. (1) involves the terms $O(\frac{\partial^2 \hat{Z}}{\partial\lambda\partial r})$, $O(\frac{\partial \hat{Z}}{r\partial\lambda})$, $O(\frac{\partial \hat{Z}}{\lambda\partial r})$, $O(\frac{\hat{Z}}{\lambda r})$ [20]. It is noted that the advection contributions are accurately accounted for in the equation of $\hat{Z}$, but it is not closed due to the dissipation and pressure terms. Using Polyakov's OPE approach, Yakhot has shown that the dissipation term can be treated easily while the pressure term has an additional difficulty. The pressure contribution leads to effective energy redistribution between components of the velocity field, and it has a nontrivial effect in the dynamics of the NS equation. Proceeding to find a closed equation for the generating function of the longitudinal velocity difference, $\hat{Z}$, the dissipation and pressure terms in Eq. (1) give contributions, and the longitudinal part of the dissipation term renormalizes the coefficient in front of $O(\frac{1}{\lambda})$ in the equation for $\hat{Z}$ [13]. Also, it generates a term with the order of $O(U)$ which can be written in terms of $\hat{Z}$ as $\lambda\frac{\partial \hat{Z}}{\partial\lambda}$. Taking into account all the possible terms and using the symmetry of the PDF, i.e., $P(U,r) = P(-U,-r)$, the following closed equation for $\hat{Z}$ can be found [13]:

$$\frac{\partial^2 \hat{Z}}{\partial\lambda\partial r} - \frac{B_0}{\lambda}\frac{\partial \hat{Z}}{\partial r} = \frac{A}{r}\frac{\partial \hat{Z}}{\partial\lambda} - C\lambda\frac{\partial \hat{Z}}{\partial\lambda} + 3r^2\lambda^2\hat{Z}, \quad (3)$$

where the parameters A, B_0, and C are determined from the theory. Also we suppose that $k_{\mu\nu}$ has the structure $k_{\mu\nu}(\mathbf{r}_{i,j}) = k(0)[1 - \frac{|\mathbf{r}_{i,j}|^2}{2L^2}\delta_{\mu,\nu} - \frac{(\mathbf{r}_{i,j})_\mu(\mathbf{r}_{i,j})_\nu}{L^2}]$ with $k(0) = 1$ and $\mathbf{r}_{i,j} = \mathbf{x}_i - \mathbf{x}_j$. The Gaussian assumption for "single-point" probability density fixes the value of the coefficient $C = \frac{u_{\rm rms}}{L}$ and the C term corresponds to the breakdown of G invariance in the limited Polyakov's sense [1].

In the limit $r \to 0$ the equation for the probability density is derived from Eq. (3) as

$$-\frac{\partial}{\partial U}U\frac{\partial P}{\partial r} - B_0\frac{\partial P}{\partial r} = -\frac{A}{r}\frac{\partial}{\partial U}UP + \frac{u_{\rm rms}}{L}\frac{\partial^2}{\partial U^2}UP. \quad (4)$$

Using the exact results $S_3 = -\frac{4}{5}\epsilon r$ in the small scale (ϵ is the mean energy dissipation rate) one finds $A = \frac{3+B}{3}$, where $B = -B_0 > 0$ [13]. It is easy to see that Eq. (4) can be written as $\partial_r P = (-\partial_U U - B_0)^{-1} \times [-(A/r)\partial_U U + (u_{\rm rms}/L)\partial_U^2 U]P$, and so its solution can obviously be written as a scalar-ordered exponential [23],

$$P(U,r) = \mathcal{T}[e_+^{\int_{r_0}^{r} dr' L_{\rm KM}(U,r')} P(U,r_0)],$$

where $L_{\rm KM}$ can be obtained formally by computing the inverse operator. Using the properties of scale-ordered exponentials the conditional probability density will satisfy the Chapman-Kolmogorov equation. Equivalently we derive that the probability density and, as a result, the conditional probability density of velocity increments satisfy a KM evolution equation,

$$-\frac{\partial P}{\partial r} = \sum_{n=1}^{\infty}(-1)^n\frac{\partial^n}{\partial U^n}[D^{(n)}(r,U)P], \quad (5)$$

where $D^{(n)}(r,U) = \frac{\alpha_n}{r}U^n + \beta_n U^{n-1}$. We have found that the coefficients α_n and β_n depend on A, B, $u_{\rm rms}$, and the integral length scale L which are given by the recursion relations $-\sum_{n=1}^{m}\frac{m!}{(m-n)!}\alpha_n = \frac{Am}{m+B}$, $-\sum_{n=1}^{m}\frac{m!}{(m-n)!}\beta_n = (\frac{u_{\rm rms}}{L})\frac{m(m-1)}{(m+B)}$. We scale the velocities as $\tilde{U} = \frac{U}{(r/L)^{1/3}}$ and introduce a logarithmic length scale $\lambda = \ln(\frac{L}{r})$ which varies from zero to infinity as r decreases from L to η. Thus the form of $\tilde{D}^{(1)}(\tilde{U}, r)$ and $\tilde{D}^{(2)}(\tilde{U}, r)$ in the equivalent description would be $\tilde{D}^{(1)}(\tilde{U}, r) = -(\frac{A}{1+B})\tilde{U}$ and $\tilde{D}^{(2)}(\tilde{U}, r) = (\frac{A}{(2+B)(1+B)})\tilde{U}^2 - (\frac{r}{L})^{2/3} u_{\rm rms}(\frac{(1)}{(2+B)})\tilde{U}$.

The drift and diffusion coefficients for various scales of λ, determined in the theory of Yakhot, show the same functional form as the calculated coefficients from the experimental data [21,22].

In comparison with the phenomenological theory of Friedrich and Pienke we are able to construct a KM equation for velocity increments that is analytically derived from the Yakhot theory which is based on just general underlying symmetries and OPE conjecture. Furthermore, this viewpoint on Eq. (4) gives the expressions for scale dependence of the coefficients in the KM equation. The important result is that scale dependent KM coefficients are proportional to $u_{\rm rms}$ which gives a probable relationship between the breakdown of G invariance and the scale dependence of the KM coefficients in the equivalent theory. The two unknown parameters A and B in the theory are reduced to 1 by fitting the $\xi_3 = 1$, so all the scaling exponents and $D^{(n)}$'s are described by one parameter, B. Considering the results in [13,21] on which the value of B is obtained, we have used the value $B \cong 20$ and have calculated the numerical values of the KM coefficients. Ratios of the first three coefficients α_n and β_n are $\alpha_3/\alpha_2 = 0.04$, $\alpha_4/\alpha_2 = 0.001$, $\beta_3/\beta_2 = 0.04$, and $\beta_4/\beta_2 = 0.001$. From the comparison of numerical values of higher order coefficients we find that the series can be cut safely after the second term, and a good approximation for the evolution operator of velocity increments is a Fokker-Planck operator. According to [13] the value of the parameter, $B \cong 20$ is calculated numerically in the limit of infinite Reynolds numbers. Using this value for the calculation of the numerical values of $\tilde{D}^{(1)}$ and $\tilde{D}^{(2)}$ we find that the contribution of scale dependent terms is essentially negligible. As it is well known, the Fokker-Planck description of probability measure is equivalent with the Langevin description written as [23], $\frac{\partial \tilde{U}}{\partial \lambda} = \tilde{D}^{(1)}(\tilde{U}, \lambda) + \sqrt{\tilde{D}^{(2)}(\tilde{U}, \lambda)}\,\eta(\lambda)$, where $\eta(\lambda)$ is a white noise and the diffusion term acts as a multiplicative noise. By considering the Ito prescription and using the path-integral representation of the Fokker-Planck equation, we can give an expression for all the possible paths in the configuration space of velocity differences and thus demonstrate the change of the measure under the change of scale, i.e.,

$$P(\tilde{U}_2, \lambda_2 \,|\, \tilde{U}_1, \lambda_1) = \int \mathcal{D}[\tilde{U}] e^{-\int_{\lambda_1}^{\lambda_2} d\lambda [\frac{\partial \tilde{U}}{\partial \lambda} - \tilde{D}^{(1)}(\tilde{U}, \lambda)]^2 / 4\tilde{D}^{(2)}(\tilde{U}, \lambda)}. \quad (6)$$

When calculating, the measure of the path integral is meaningful when some form of discretization is chosen [23], but we have written it in a formal way. Using the forms of $\tilde{D}^1$ and $\tilde{D}^2$ and approximating them with scale independent ones in the infinite Reynolds number limit, one can easily see that the transition functional can be written in terms of $\ln \tilde{U}$. It is an easy way to see how the large scale $\lambda \to 0$ Gaussian probability density can change its shape when going to small scales $\lambda \to \infty$ and consequently give rise to intermittent behavior. Instead of working with the probability functional of velocity increments, the formal solution of Fokker-Planck equation, as a scale-ordered exponential [24], can be converted to an integral representation for the probability measure of velocity increments when $\tilde{D}^1 \cong -\alpha_1(\lambda)\tilde{U}$ and $\tilde{D}^2 \cong \alpha_2(\lambda)\tilde{U}^2$, i.e.,

$$P(\tilde{U}, \lambda) = \frac{e^{\gamma_0(\lambda)}}{\sqrt{4\pi\gamma(\lambda)}} \int_{-\infty}^{+\infty} e^{-(s^2)/[4\gamma(\lambda)]} \phi(\tilde{U} e^{\gamma_1(\lambda) - s})\, ds\,, \quad (7)$$

where $\gamma_0(\lambda) = \int_0^\lambda [-\alpha_1(\lambda') + 2\alpha_2(\lambda')]\, d\lambda'$ and $\gamma_1(\lambda) = \int_0^\lambda [-\alpha_1(\lambda') + 3\alpha_2(\lambda')]\, d\lambda'$ and $\gamma(\lambda) = \int_0^\lambda \alpha_2(\lambda')\, d\lambda'$ and $\phi(\tilde{U})$ is the probability measure in the integral length scales $(\lambda \to 0)$. We consider the Gaussian distribution $\phi(\tilde{U}) \cong e^{-m\tilde{U}^2}$ in the integral scale which is a reasonable choice (experimental data show that up to third moments the PDF in the integral scale is consistent with the Gaussian distribution [13]), and we derive the dependence of the variance of the probability density on the scale in the limit when the original distribution satisfies the condition $m \ll 1$. The result shows an exponential dependence such as $m \to m e^{2\zeta}$, where $\zeta = 3\alpha_2 - \alpha_1$. The consistent picture with the shape change of probability measure under the scale is that when λ grows, the width decreases and vice versa. Moreover, we should emphasize that the shape change is somehow complex which gives some corrections in order $O(m^2\tilde{U}^4)$ even in this simplifying limit, i.e., $m \ll 1$. Starting with a Gaussian measure at integral scales and using the calculated scale independent Fokker-Planck coefficients, we have numerically calculated the PDF's for fully developed turbulence and Burgers turbulence in different length scales from which their plots in Figs. 1 and 2 are completely compatible with experimental and simulation results [13,21,22]. The extreme case of Burgers problem (i.e., $B \cong 0$) shows the ever localizing behavior as if in the limit of $\lambda \to \infty$ goes to a Dirac delta function which again is consistent with our knowledge about Burgers problem [6,13]. Clearly Eqs. (4) and (5) give the same result for the multifractal exponent of structure function, i.e., $S_n(r) \cong A_n r^{\xi_n}$ is derived to be $\xi_n = \frac{(3+B)n}{3(n+B)}$ [13].

In summary, we have constructed a theoretical bridge between two recent theories involving the statistics of longitudinal velocity increment in fully developed turbulence. On the basis of the recent theory proposed by Yakhot we showed that the probability density of longitudinal velocity components satisfies a Kramers-Moyal equation which encodes the Markovian property of these fluctuations in a necessary way. We are able to give the exact form of Kramers-Moyal coefficients in terms of a basic parameter in the Yakhot theory B. The qualitative behavior of drift and diffusion terms are consistent with the experimental

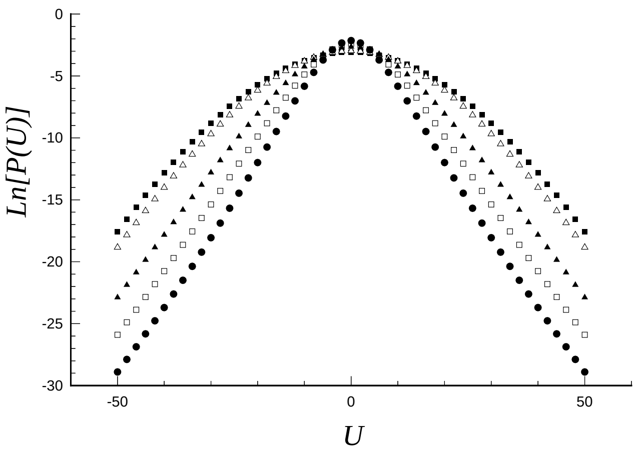

FIG. 1. Schematic view of the logarithm of PDF in terms of different length scales. These graphs are numerically obtained from the integral representation of PDF at the Fokker-Planck approximation. The curves correspond with the scales $L/r = 1.5$, 2, 5, 10, and 20.

outcomes [21]. As the most prominent result of our work, we could find the form of path probability functional of the velocity increments in scale which naturally encodes the scale dependence of probability density. This gives a clear picture about the intermittent nature in fully developed turbulence.

We should emphasize that the derivation of the KM equation is not restricted to Polyakov's specific approach. One can show that similar results could be obtained by the conditional averaging methods [25,26]. A clearly analytic form of the KM coefficients $D^{(n)}$ can be estimated numerically but analytic derivation is not possible [26]. Our work might be generalized to give a theoretical basis for the Markovian fluctuations of the moments of height difference in the surface growth problems like KPZ [16,27], and we believe that it would be possible to derive the Kramers-Moyal description for the statistics of energy dissipation [28].

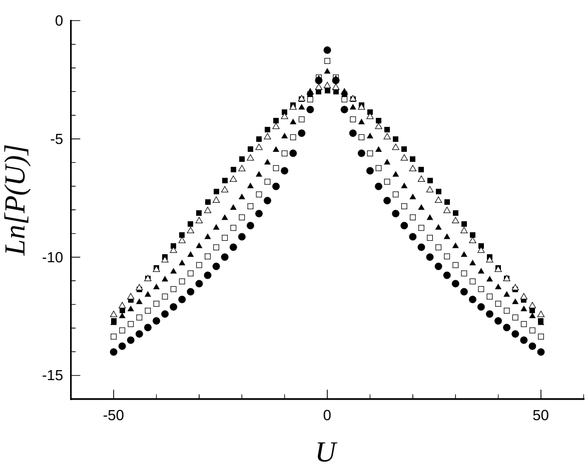

FIG. 2. Schematic view of the logarithm of PDF in the Burgers turbulence ($B \cong 0$), in terms of different length scales. These graphs are numerically obtained from the integral representation of PDF at the Fokker-Planck approximation. The scales are $L/r = 1.5$, 2, 5, 10, and 20.

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