General Physics I

Chapter 3

Sharif University of Technology
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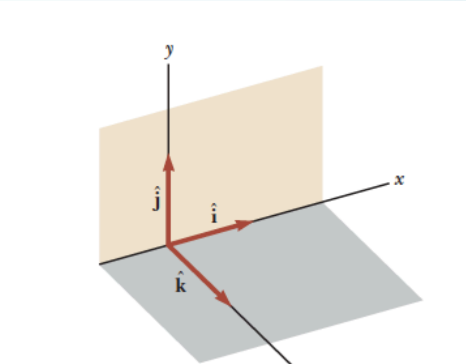
M. Reza Rahimi Tabar

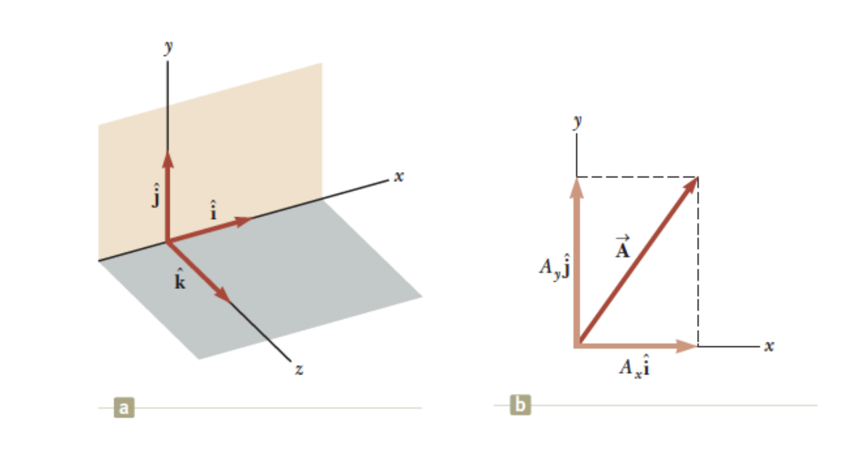
Chapter 3
Vectors

- Used to describe the position of a point in space
- Coordinate system (frame) consists of
  - a fixed reference point called the origin
  - specific axes with scales and labels
  - instructions on how to label a point relative to the origin and the axes



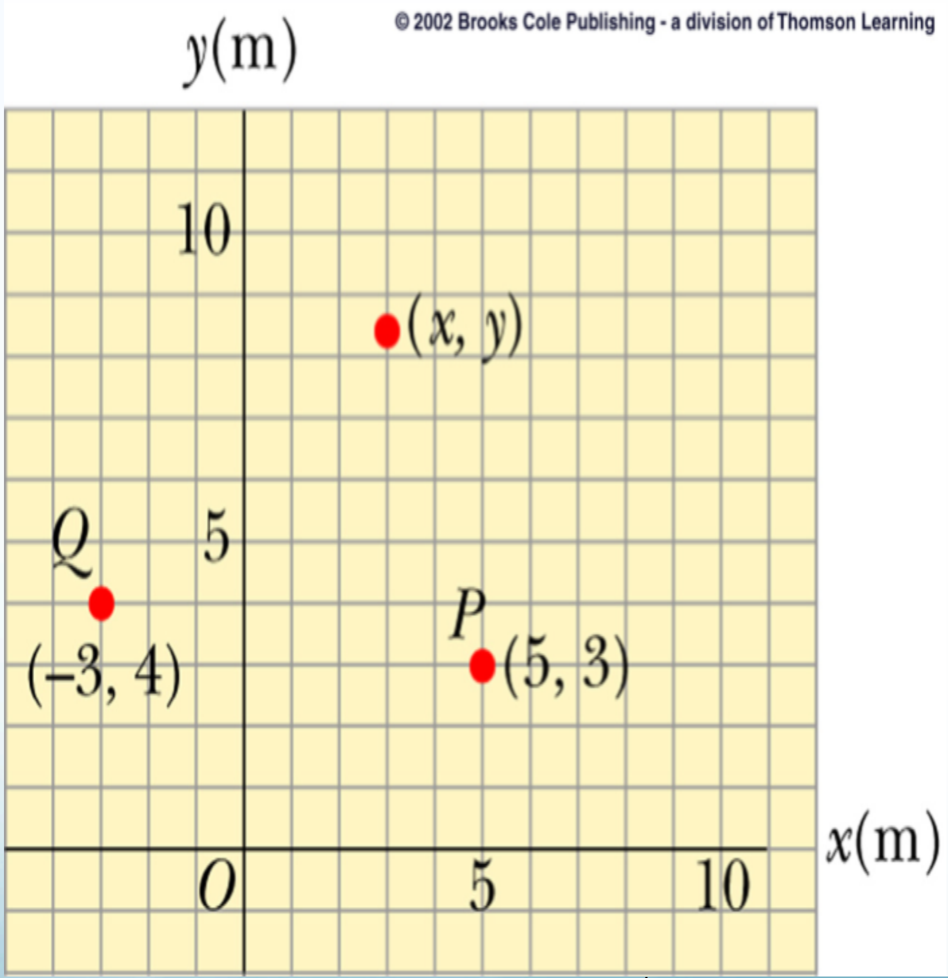
Types of Coordinate Systems (2D)

- Cartesian (1D, 2D, 3D,....)
- Plane polar (2D)
- Cylindrical coordinate (3D,..)
- Spherical coordinate (3D,..)
- etc,



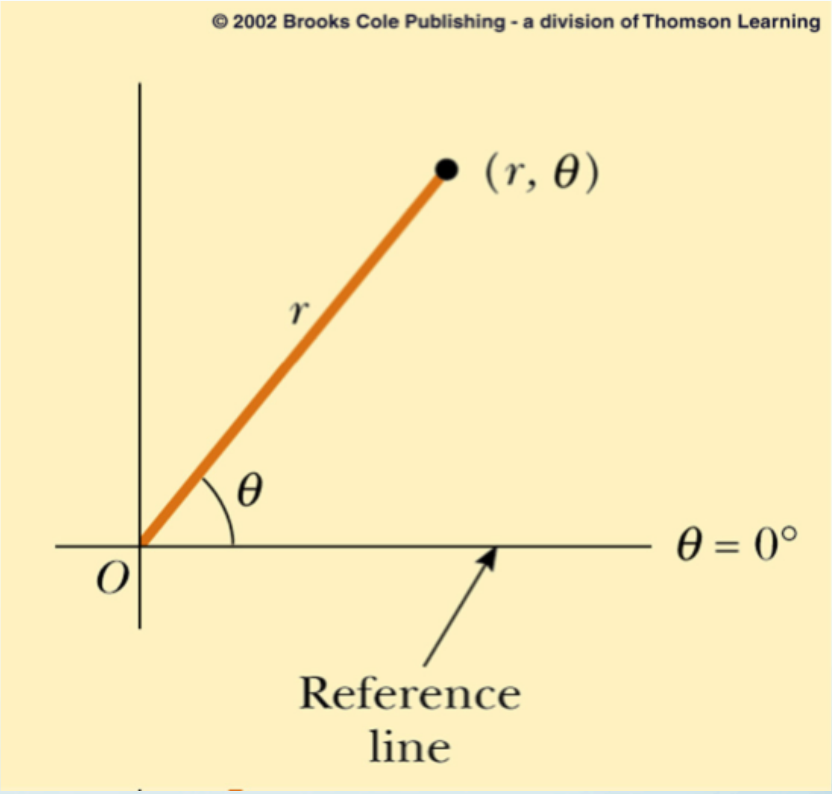
Cartesian coordinate system

- also called rectangular coordinate system
- x- and y- axes
- points are labeled (x,y)



Plane polar coordinate system

- origin and reference line are noted
- point is distance r from the origin in the direction of angle θ, from reference line
- points are labeled (r,θ)



Scalar and Vector Quantities

- Scalar quantities are completely described by magnitude only (temperature, length,…)
- Vector quantities need both magnitude (size) and direction to completely describe them (force, displacement, velocity,…)
  - Represented by an arrow, the length of the arrow is proportional to the magnitude of the vector
  - Head of the arrow represents the direction

Vector Notation

- When handwritten, use an arrow: $\vec{A}$
- When printed, will be in bold print: **A**
- When dealing with just the magnitude of a vector in print, an italic letter will be used: A

Properties of Vectors

- Equality of Two Vectors
  - Two vectors are **equal** if they have the same magnitude and the same direction
- Movement of vectors in a diagram
  - Any vector can be moved parallel to itself without being affected

More Properties of Vectors

- Negative Vectors
  - Two vectors are **negative** if they have the same magnitude but are 180° apart (opposite directions)
    - **A** = -**B**
- Resultant Vector
  - The **resultant** vector is the sum of a given set of vectors

Adding Vectors

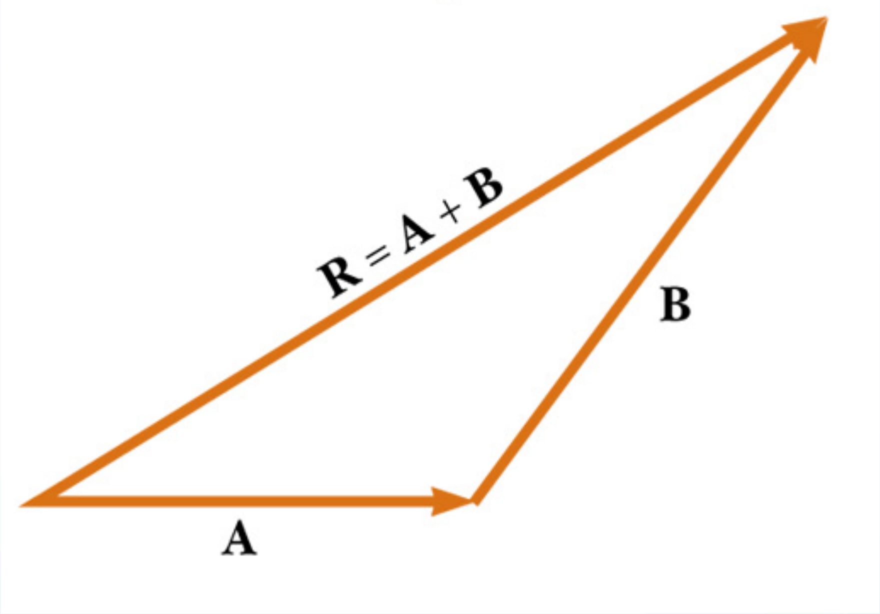
- When adding vectors, their directions must be taken into account
- Units must be the same
- Graphical Methods
  - Use scale drawings
- Algebraic Methods
  - More convenient

Adding Vectors Graphically (Triangle or Polygon Method)

- Choose a scale
- Draw the first vector with the appropriate length and in the direction specified, with respect to a coordinate system
- Draw the next vector with the appropriate length and in the direction specified, with respect to a coordinate system whose origin is the end of vector **A** and parallel to the coordinate system used for **A**

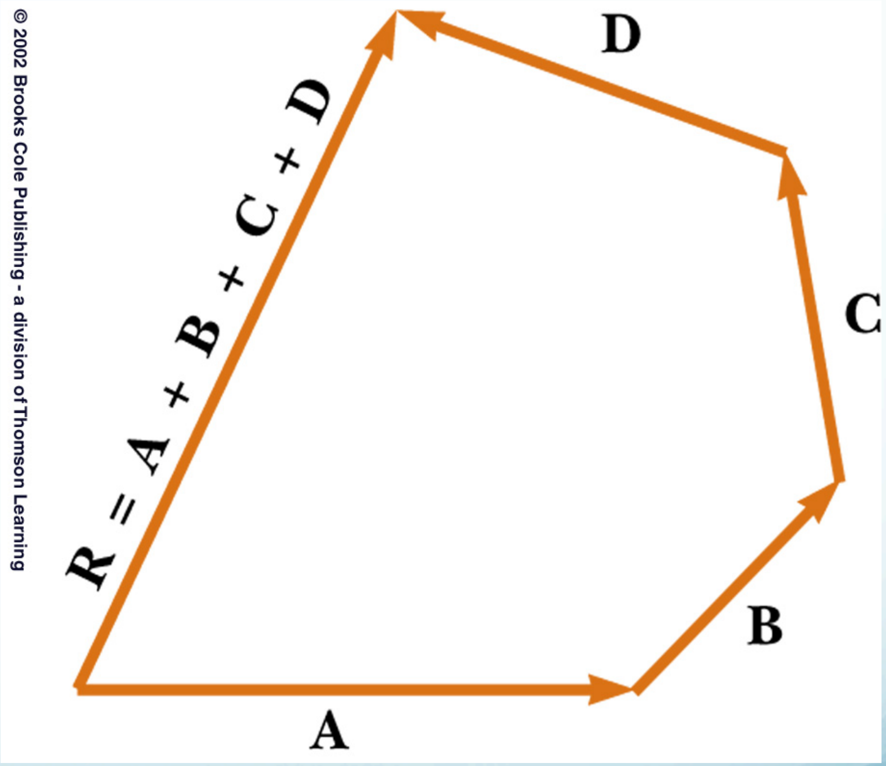
Graphically Adding Vectors

- Continue drawing the vectors “tip-to-tail”
- The resultant is drawn from the origin of **A** to the end of the last vector
- Measure the length of **R** and its angle
  - Use the scale factor to convert length to actual magnitude



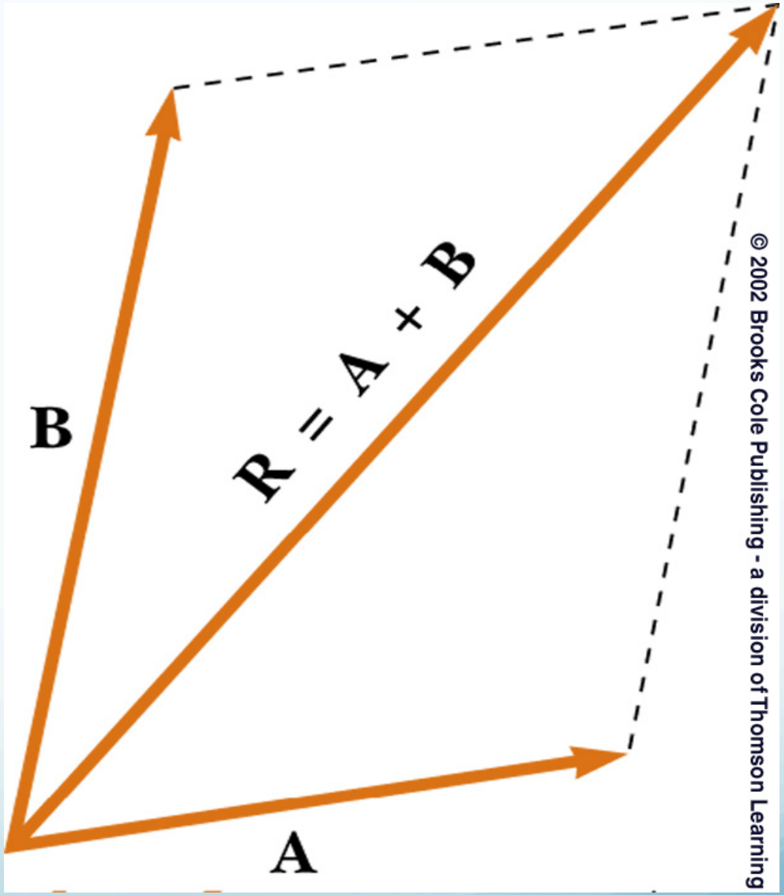
Graphically Adding Vectors

- When you have many vectors, just keep repeating the process until all are included
- The resultant is still drawn from the origin of the first vector to the end of the last vector



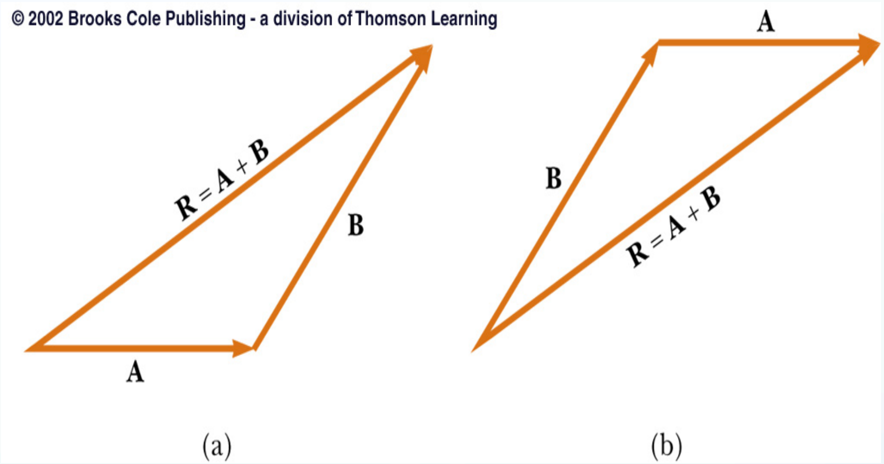
Alternative Graphical Method

- When you have only two vectors, you may use the **Parallelogram Method**
- All vectors, including the resultant, are drawn from a common origin
  - The remaining sides of the parallelogram are sketched to determine the diagonal, **R**



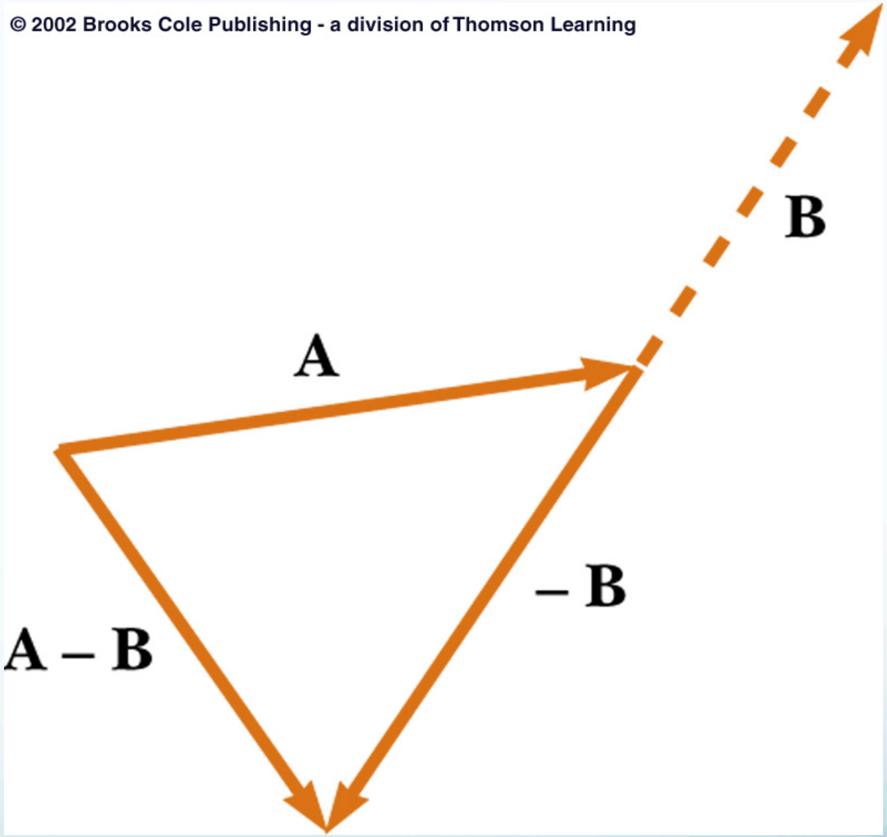
Notes about Vector Addition

- Vectors obey the **Commutative Law of Addition**
  - The order in which the vectors are added doesn't affect the result



Vector Subtraction

- Special case of vector addition
- If **A** – **B**, then use **A**+(**-B**)
- Continue with standard vector addition procedure

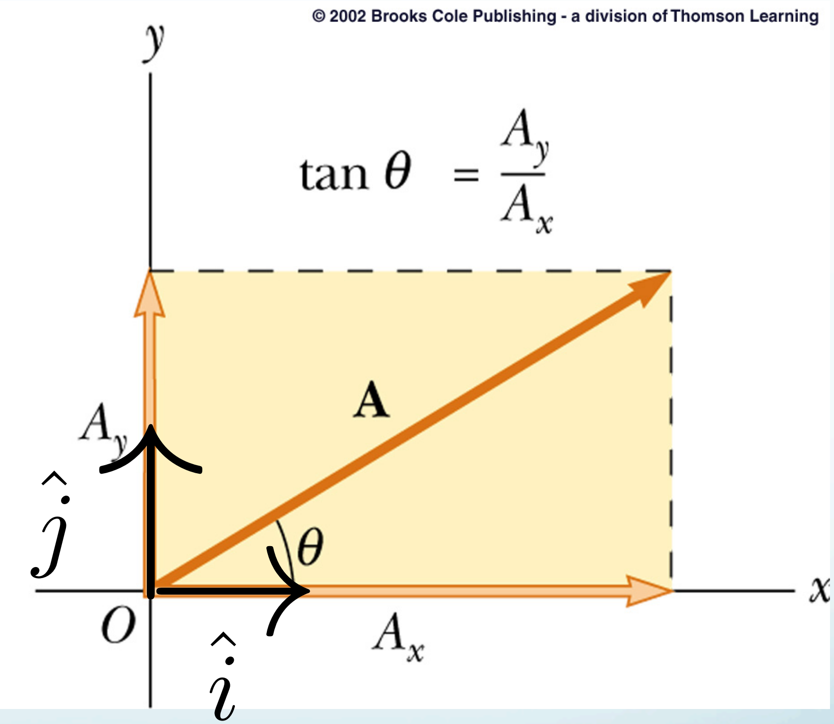


Multiplying or Dividing a Vector by a Scalar

- The result of the multiplication or division is a vector
- The magnitude of the vector is multiplied or divided by the scalar
- If the scalar is positive, the direction of the result is the same as of the original vector
- If the scalar is negative, the direction of the result is opposite that of the original vector

Components of a Vector

- A **component** is a part
- It is useful to use **rectangular components**
  - These are the projections of the vector along the x- and y-axes



$$A_x = A\cos\theta$$

$$A_y = A\sin\theta$$

Components of a Vector

- The x-component of a vector is the projection along the x-axis

$$A_x = A\cos\theta \qquad \vec{A}_x = \hat{i}A_x$$

- The y-component of a vector is the projection along the y-axis

$$A_y = A\sin\theta \qquad \vec{A}_y = \hat{j}A_y$$

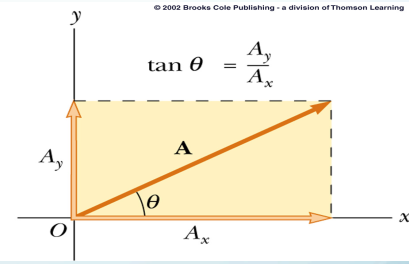
- Then,

$$\vec{\mathbf{A}} = \vec{A}_x + \vec{A}_y$$

More About Components of a Vector

- The previous equations are valid **only if θ is measured with respect to the x-axis**
- The components can be positive or negative and will have the same units as the original vector
- The components are the legs of the right triangle whose hypotenuse is **A**

$$A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{A_y}{A_x}$$



  - May still have to find *θ* with respect to the positive x-axis

Adding Vectors Algebraically

- Choose a coordinate system and sketch the vectors
- Find the x- and y-components of all the vectors
- Add all the x-components
  - This gives R_x:

$$R_x = \sum A_x$$

Adding Vectors Algebraically

- Add all the y-components
  - This gives R_y:

$$R_y = \sum A_y$$

- Use the Pythagorean Theorem to find the magnitude of the Resultant:

$$R = \sqrt{R_x^2 + R_y^2}$$

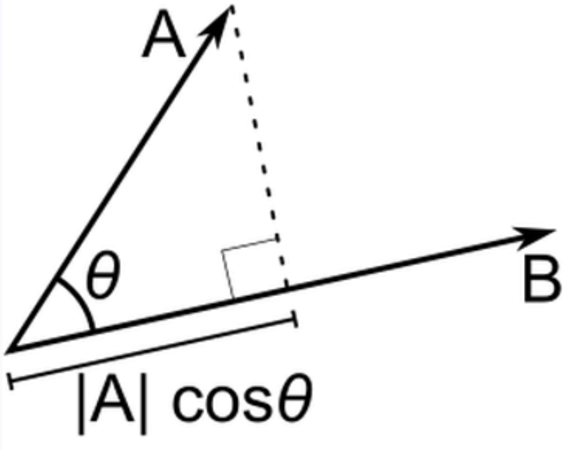
- Use the inverse tangent function to find the direction of R:

$$\theta = \tan^{-1} \frac{R_y}{R_x}$$

Products of Vectors

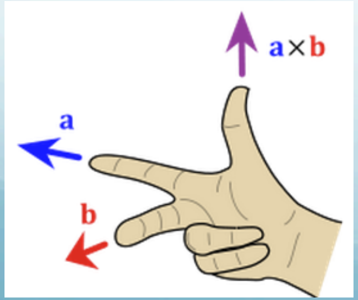
- Inner product

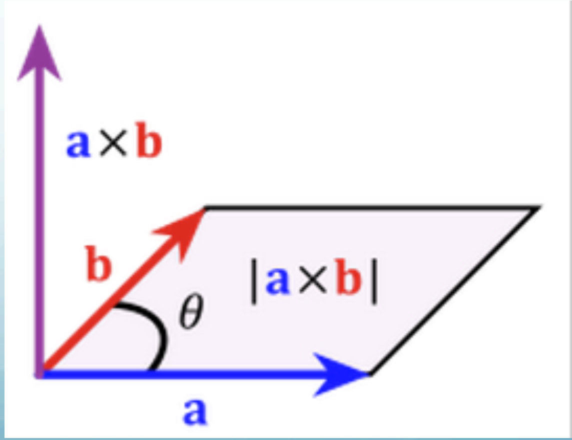
$$A \cdot B = |A| * |B| * \cos(\Theta)$$



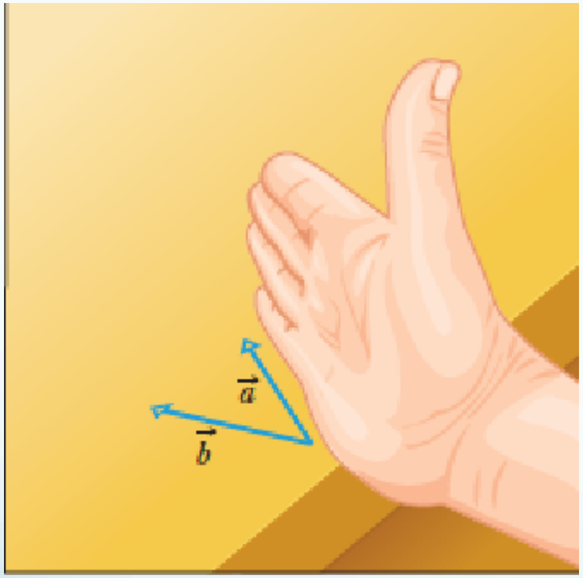
- Cross product

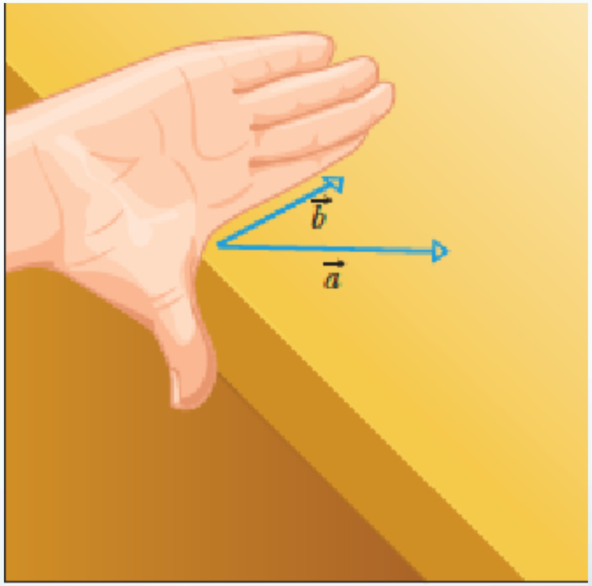
$$A = \|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta.$$





example





In terms of components

Definition

We define the *dot product* of two vectors

$\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{w} = c\mathbf{i} + d\mathbf{j}$

to be

$\mathbf{v} \cdot \mathbf{w} = ac + bd$

Dot Product in $\mathbf{R}^3$

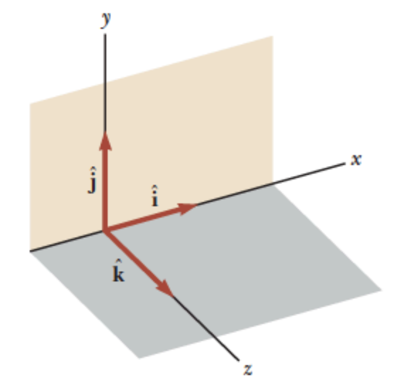
If

$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{w} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$

then

$\mathbf{v} \cdot \mathbf{w} = ad + be + cf$

$$i.i = 1$$
$$i.j = 0$$
$$\ldots$$
$$i \times j = k$$
$$j \times k = i$$
$$i \times k = ?$$



Definition

Let **u** = a**i** + b**j** + c**k** and **v** = d**i** + e**j** + f**k** be vectors then we define the *cross product* **v** x **w** by the determinant of the matrix:

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$$

$$\begin{pmatrix} b & c \\ e & f \end{pmatrix}\mathbf{i} - \begin{pmatrix} a & c \\ d & f \end{pmatrix}\mathbf{j} + \begin{pmatrix} a & b \\ d & e \end{pmatrix}\mathbf{k}$$

$$= (bf - ce)\,\mathbf{i} + (cd - af)\,\mathbf{j} + (ae - bd)\,\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$$

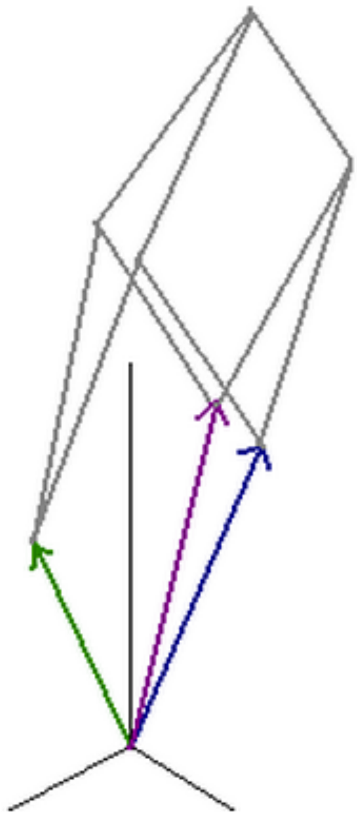
Triple product

To find the volume of the parallelepiped spanned by three vectors **u**, **v**, and **w**, we find the triple product:

Volume = u · (v x w)

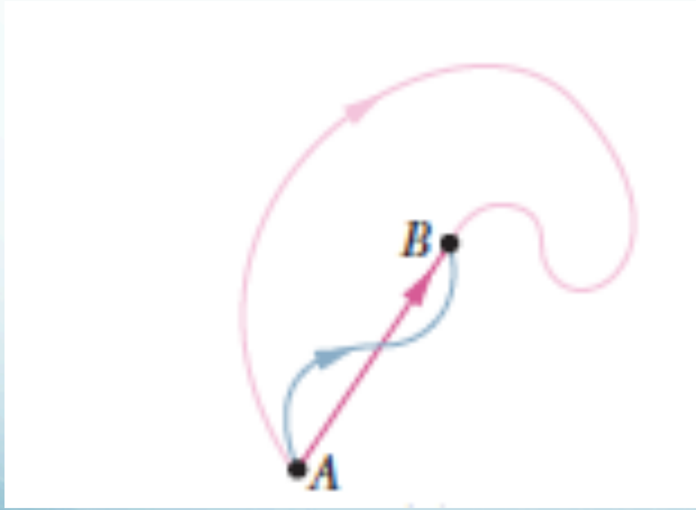
This can be found by computing the determinate of the three vectors:

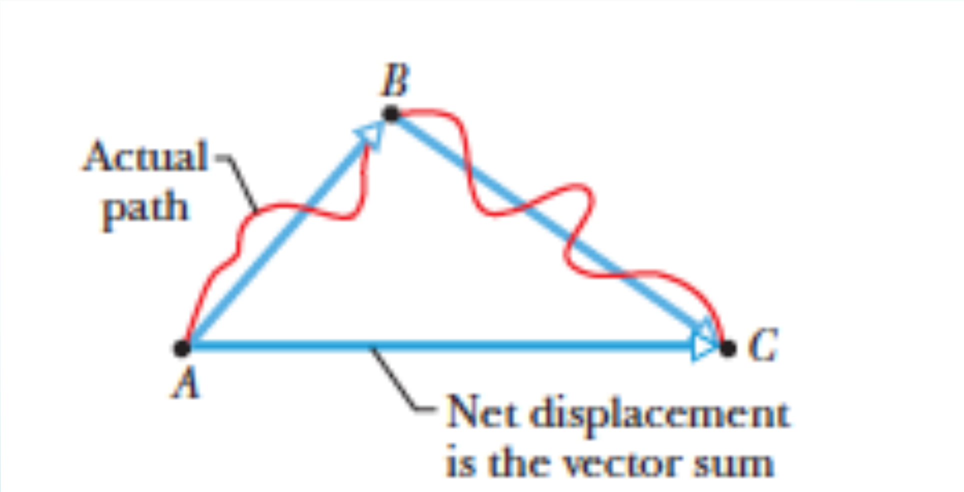
$$\left| \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \right|$$



Examples:

- Displacement vector





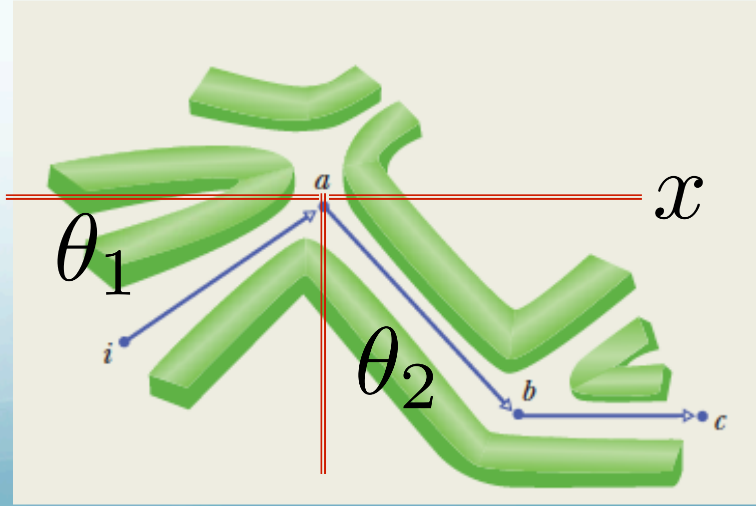
Sample Problem 3.03

Searching through a hedge maze

A hedge maze is a maze formed by tall rows of hedge. After entering, you search for the center point and then for the exit. Figure 3-16*a* shows the entrance to such a maze and the first two choices we make at the junctions we encounter in moving from point *i* to point *c*. We undergo three displacements as indicated in the overhead view of Fig. 3-16*b*:

$$d_1 = 6.00\text{ m} \qquad \theta_1 = 40°$$
$$d_2 = 8.00\text{ m} \qquad \theta_2 = 30°$$
$$d_3 = 5.00\text{ m} \qquad \theta_3 = 0°,$$

where the last segment is parallel to the superimposed *x* axis. When we reach point *c*, what are the magnitude and angle of our net displacement $\vec{d}_{net}$ from point *i*?



KEY IDEAS

(1) To find the net displacement $\vec{d}_{net}$, we need to sum the three individual displacement vectors:

$$\vec{d}_{net} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3.$$

(2) To do this, we first evaluate this sum for the x components alone,

$$d_{net,x} = d_{1x} + d_{2x} + d_{3x}, \tag{3-16}$$

and then the y components alone,

$$d_{net,y} = d_{1y} + d_{2y} + d_{3y}. \tag{3-17}$$

(3) Finally, we construct $\vec{d}_{net}$ from its x and y components.

$$d_{1x} = (6.00\text{ m})\cos 40^\circ = 4.60\text{ m}$$
$$d_{2x} = (8.00\text{ m})\cos(-60^\circ) = 4.00\text{ m}$$
$$d_{3x} = (5.00\text{ m})\cos 0^\circ = 5.00\text{ m}.$$

Equation 3-16 then gives us

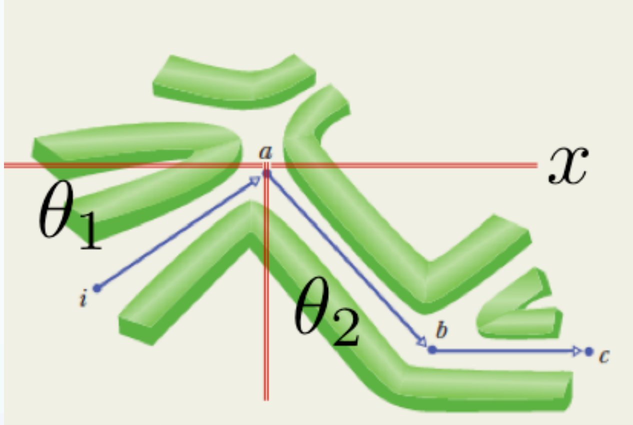
$$d_{\text{net},x} = +4.60\text{ m} + 4.00\text{ m} + 5.00\text{ m}$$
$$= 13.60\text{ m}.$$

Similarly, to evaluate Eq. 3-17, we apply the y part of Eq. 3-5 to each displacement:

$$d_{1y} = (6.00\text{ m})\sin 40^\circ = 3.86\text{ m}$$
$$d_{2y} = (8.00\text{ m})\sin(-60^\circ) = -6.93\text{ m}$$
$$d_{3y} = (5.00\text{ m})\sin 0^\circ = 0\text{ m}.$$

Equation 3-17 then gives us

$$d_{\text{net},y} = +3.86\text{ m} - 6.93\text{ m} + 0\text{ m}$$
$$= -3.07\text{ m}.$$



the vector forms the hypotenuse. We find the magnitude and angle of $\vec{d}_{net}$ with Eq. 3-6. The magnitude is

$$d_{net} = \sqrt{d_{net,x}^2 + d_{net,y}^2} \tag{3-18}$$

$$= \sqrt{(13.60\text{ m})^2 + (-3.07\text{ m})^2} = 13.9\text{ m}. \quad \text{(Answer)}$$

To find the angle (measured from the positive direction of x), we take an inverse tangent:

$$\theta = \tan^{-1}\left(\frac{d_{net,y}}{d_{net,x}}\right) \tag{3-19}$$

$$= \tan^{-1}\left(\frac{-3.07\text{ m}}{13.60\text{ m}}\right) = -12.7°. \quad \text{(Answer)}$$

The angle is negative because it is measured clockwise from

Sample Problem 3.04

Adding vectors, unit-vector components

$$\vec{a} = (4.2\text{ m})\hat{\text{i}} - (1.5\text{ m})\hat{\text{j}},$$
$$\vec{b} = (-1.6\text{ m})\hat{\text{i}} + (2.9\text{ m})\hat{\text{j}},$$
$$\vec{c} = (-3.7\text{ m})\hat{\text{j}}.$$

Sample Problem 3.04
Adding vectors, unit-vector components

$$\vec{a} = (4.2\text{ m})\hat{\text{i}} - (1.5\text{ m})\hat{\text{j}},$$
$$\vec{b} = (-1.6\text{ m})\hat{\text{i}} + (2.9\text{ m})\hat{\text{j}},$$
$$\vec{c} = (-3.7\text{ m})\hat{\text{j}}.$$

$$r = \sqrt{(2.6\text{ m})^2 + (-2.3\text{ m})^2} \approx 3.5\text{ m} \qquad \text{(Answer)}$$

and the angle (measured from the $+x$ direction) is

$$\theta = \tan^{-1}\left(\frac{-2.3\text{ m}}{2.6\text{ m}}\right) = -41^\circ, \qquad \text{(Answer)}$$

Sample Problem 3.05
Angle between two vectors using dot products

What is the angle ϕ between $\vec{a} = 3.0\hat{i} - 4.0\hat{j}$ and $\vec{b} = -2.0\hat{i} + 3.0\hat{k}$?

The angle between the directions of two vectors is included in the definition of their scalar product (Eq. 3-20):

$$\vec{a} \cdot \vec{b} = ab \cos \phi. \tag{3-28}$$

Calculations: In Eq. 3-28, a is the magnitude of $\vec{a}$, or

$$a = \sqrt{3.0^2 + (-4.0)^2} = 5.00, \tag{3-29}$$

and b is the magnitude of $\vec{b}$, or

$$b = \sqrt{(-2.0)^2 + 3.0^2} = 3.61. \tag{3-30}$$

We can separately evaluate the left side of Eq. 3-28 by writing the vectors in unit-vector notation and using the distributive law:

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (3.0\hat{i} - 4.0\hat{j})\cdot(-2.0\hat{i} + 3.0\hat{k}) \\ &= (3.0\hat{i})\cdot(-2.0\hat{i}) + (3.0\hat{i})\cdot(3.0\hat{k}) \\ &\quad + (-4.0\hat{j})\cdot(-2.0\hat{i}) + (-4.0\hat{j})\cdot(3.0\hat{k}).\end{aligned}$$

We next apply Eq. 3-20 to each term in this last expression. The angle between the unit vectors in the first term ($\hat{i}$ and $\hat{i}$) is 0°, and in the other terms it is 90°. We then have

$$\begin{aligned}\vec{a}\cdot\vec{b} &= -(6.0)(1) + (9.0)(0) + (8.0)(0) - (12)(0) \\ &= -6.0.\end{aligned}$$

Substituting this result and the results of Eqs. 3-29 and 3-30 into Eq. 3-28 yields

$$-6.0 = (5.00)(3.61)\cos\phi,$$

so $$\phi = \cos^{-1}\frac{-6.0}{(5.00)(3.61)} = 109^\circ \approx 110^\circ. \qquad \text{(Answer)}$$

Sample Problem 3.07

Cross product, unit-vector notation

If $\vec{a} = 3\hat{i} - 4\hat{j}$ and $\vec{b} = -2\hat{i} + 3\hat{k}$, what is $\vec{c} = \vec{a} \times \vec{b}$?

Calculations: Here we write

$$\vec{c} = (3\hat{i} - 4\hat{j}) \times (-2\hat{i} + 3\hat{k})$$
$$= 3\hat{i} \times (-2\hat{i}) + 3\hat{i} \times 3\hat{k} + (-4\hat{j}) \times (-2\hat{i})$$
$$+ (-4\hat{j}) \times 3\hat{k}.$$

$$\vec{c} = -6(0) + 9(-\hat{j}) + 8(-\hat{k}) - 12\hat{i}$$
$$= -12\hat{i} - 9\hat{j} - 8\hat{k}. \qquad \text{(Answer)}$$

Additional problems

- 1

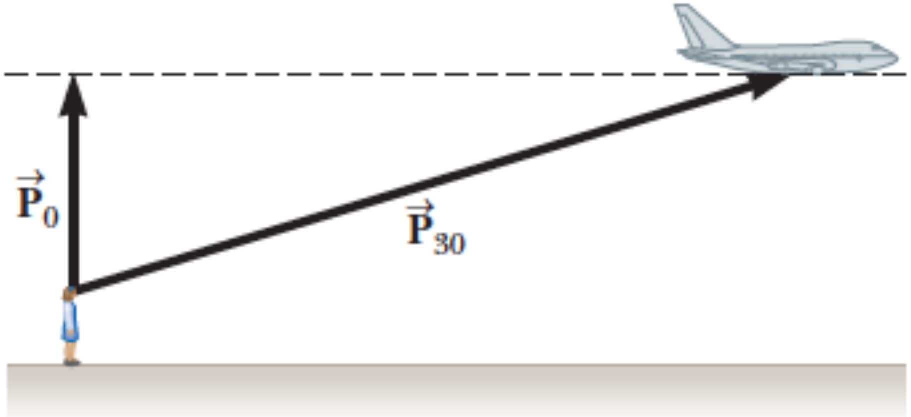
Vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ have equal magnitudes of 5.00. The sum of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is the vector $6.00\hat{\mathbf{j}}$. Determine the angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

- 2

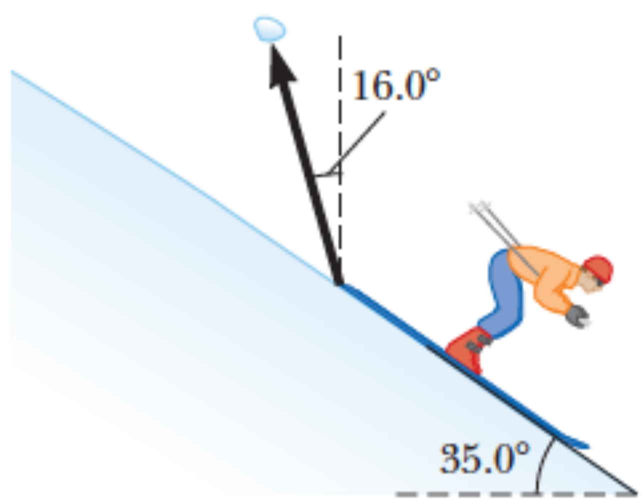
Q|C **Review.** The instantaneous position of an object is specified by its position vector leading from a fixed origin to the location of the object, modeled as a particle. Suppose for a certain object the position vector is a function of time given by $\vec{\mathbf{r}} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2t\hat{\mathbf{k}}$, where $\vec{\mathbf{r}}$ is in meters and t is in seconds. (a) Evaluate $d\vec{\mathbf{r}}/dt$. (b) What physical quantity does $d\vec{\mathbf{r}}/dt$ represent about the object?

- 3

Review. You are standing on the ground at the origin of a coordinate system. An airplane flies over you with constant velocity parallel to the *x* axis and at a fixed height of 7.60×10^3 m. At time $t = 0$, the airplane is directly above you so that the vector leading from you to it is $\vec{P}_0 = 7.60 \times 10^3\hat{j}$ m. At $t = 30.0$ s, the position vector leading from you to the airplane is $\vec{P}_{30} = (8.04 \times 10^3\hat{i} + 7.60 \times 10^3\hat{j})$ m as suggested in Figure P3.43. Determine the magnitude and orientation of the airplane's position vector at $t = 45.0$ s.



- 4. A snow-covered ski slope makes an angle of 35.0° with the horizontal. When a ski jumper plummets onto the hill, a parcel of splashed snow is thrown up to a maximum displacement of 1.50 m at 16.0° from the vertical in the uphill direction as shown in Figure P3.26. Find the components of its maximum displacement (a) parallel to the surface and (b) perpendicular to the surface.



- 5

78 What is the magnitude of $\vec{a} \times (\vec{b} \times \vec{a})$ if $a = 3.90$, $b = 2.70$, and the angle between the two vectors is 63.0°?

- 6

73 Two vectors are given by $\vec{a} = 3.0\hat{i} + 5.0\hat{j}$ and $\vec{b} = 2.0\hat{i} + 4.0\hat{j}$. Find (a) $\vec{a} \times \vec{b}$, (b) $\vec{a} \cdot \vec{b}$, (c) $(\vec{a} + \vec{b}) \cdot \vec{b}$, and (d) the component of $\vec{a}$ along the direction of $\vec{b}$.

- 7

63 Here are three vectors in meters:

$$\vec{d}_1 = -3.0\hat{\text{i}} + 3.0\hat{\text{j}} + 2.0\hat{\text{k}}$$

$$\vec{d}_2 = -2.0\hat{\text{i}} - 4.0\hat{\text{j}} + 2.0\hat{\text{k}}$$

$$\vec{d}_3 = 2.0\hat{\text{i}} + 3.0\hat{\text{j}} + 1.0\hat{\text{k}}.$$

What results from (a) $\vec{d}_1 \cdot (\vec{d}_2 + \vec{d}_3)$, (b) $\vec{d}_1 \cdot (\vec{d}_2 \times \vec{d}_3)$, and (c) $\vec{d}_1 \times (\vec{d}_2 + \vec{d}_3)$?

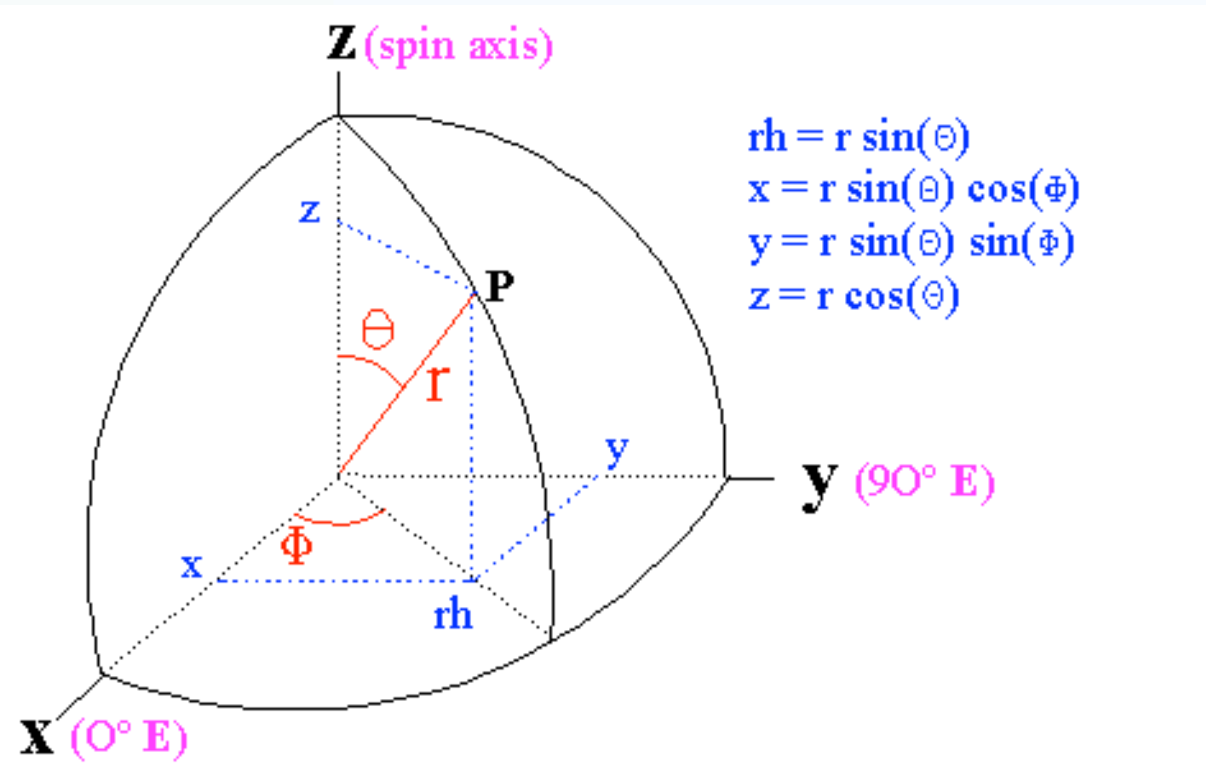
- 8

••44 GO In the product $\vec{F} = q\vec{v} \times \vec{B}$, take $q = 2$,

$$\vec{v} = 2.0\hat{i} + 4.0\hat{j} + 6.0\hat{k} \quad \text{and} \quad \vec{F} = 4.0\hat{i} - 20\hat{j} + 12\hat{k}.$$

What then is $\vec{B}$ in unit-vector notation if $B_x = B_y$?

Spherical coordinates



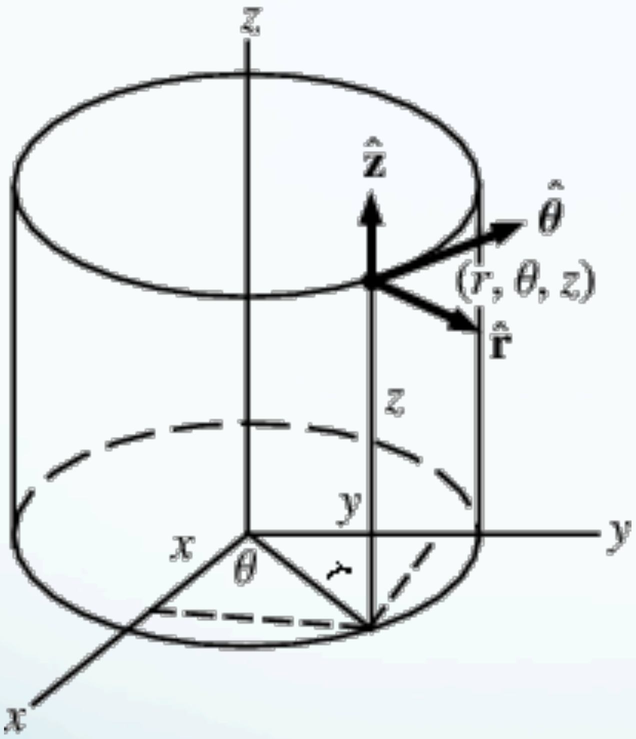
$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right)$$

where $r \in [0, \infty)$, $\phi \in [0, 2\pi)$, and $\theta \in [0, \pi]$

Cylindrical coordinate

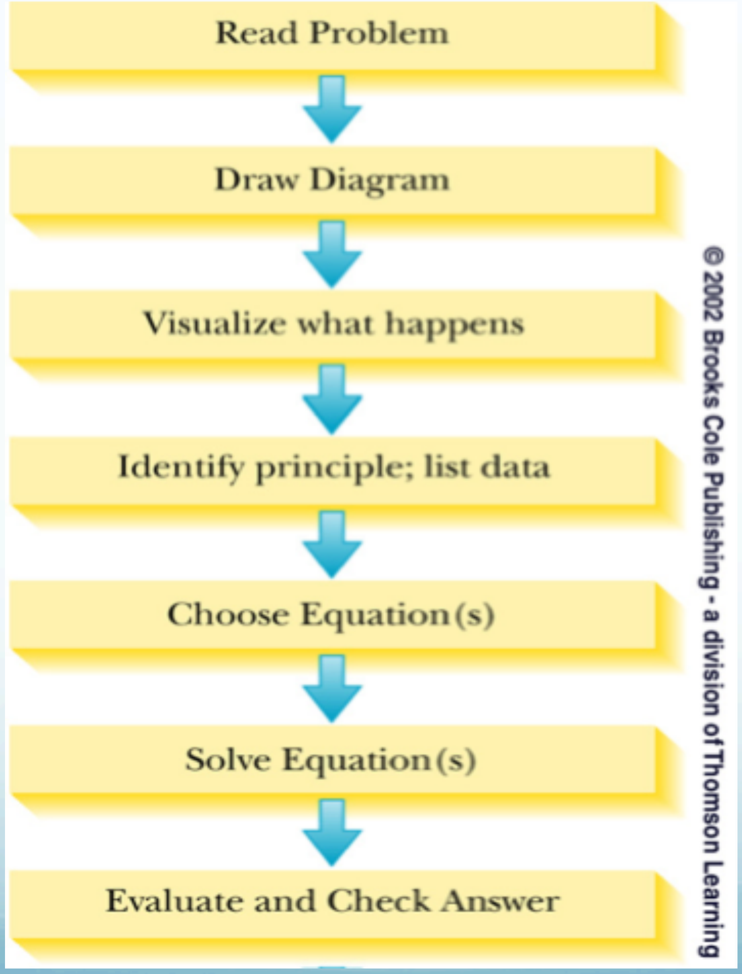


$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z.$$

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

$$r \in [0, \infty), \theta \in [0, 2\pi), z \in (-\infty, \infty),$$

Problem Solving Strategy



Problem Solving Strategy

- Read the problem
  - identify type of problem, principle involved
- Draw a diagram
  - include appropriate values and coordinate system
  - some types of problems require very specific types of diagrams

Problem Solving cont.

- Visualize the problem
- Identify information
  - identify the principle involved
  - list the data (given information)
  - indicate the unknown (what you are looking for)

Problem Solving, cont.

- Choose equation(s)
  - based on the principle, choose an equation or set of equations to apply to the problem
  - solve for the unknown
- Solve the equation(s)
  - substitute the data into the equation
  - include units

Problem Solving, final

- Evaluate the answer
  - find the numerical result
  - determine the units of the result
- Check the answer
  - are the units correct for the quantity being found?
  - does the answer seem reasonable?
    - check order of magnitude
  - are signs appropriate and meaningful?