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Exact two-point correlation functions of turbulence without pressure in three dimensions

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Abstract

We investigate exact results of isotropic turbulence in three dimensions when the pressure gradient is negligible. We derive exact two-point correlation functions of the density in three dimensions and show that the density–density correlator behaves as $|\boldsymbol{x}_1 - \boldsymbol{x}_2|^{-\alpha_3}$, where $\alpha_3 = 2 + \frac{1}{6}\sqrt{33}$. It is shown that, in three dimensions, the energy spectrum $E(k)$ in the inertial range scales with exponent $2 - \frac{1}{12}\sqrt{33} \simeq 1.5212$. We also discuss the time scale for which our exact results are valid for strong 3D turbulence in the presence of pressure. We confirm our predictions by using the recent results of numerical calculations and experiment. © 1998 Published by Elsevier Science B.V.

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1. Introduction

Recently, many efforts have been made towards a non-perturbative understanding of turbulence [1–12]. A statistical theory of turbulence has been put forward by Kolmogorov [13], and further developed by others [15–17]. The approach is to model turbulence using stochastic partial differential equations. The simplest approach to turbulence is Kolmogorov's dimensional analysis, which leads to the celebrated $k^{-5/3}$ law for the energy spectrum. This is obtained by decreeing that the energy spectrum depends neither on the wavenumber, where most of the energy resides, nor on the wavenumber of viscous dissipation. Kolmogorov conjectured that the scaling exponents are universal, independent of the statistics of large-scale fluctuation and the mechanism of the viscous damping, when the Reynolds number is sufficiently large. In fact the idea of universality is based on the notion of the "inertial subrange". By inertial subrange we mean that for very large values of the Reynolds number there is a wide separation between the scale energy input L and the typical viscous dissipation scale η at which viscous friction becomes important and the energy is turned into heat.

However, recently it has been found that there is a relation between the probability distribution function (PDF) of the velocity and those of the external force [18]. This observation has been confirmed by experiments [19], and numerical simulations [20].

In this direction, Polyakov [5] has recently offered a field theoretic method to derive the probability distribution or density of states in (1 + 1) dimensions in the problem of the randomly driven Burgers equa-

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tion [21]. In one dimension, turbulence without pressure is described by the Burgers equation (see also Ref. [14] concerning the relation between the Burgers equation and KPZ equation). In the limit of a high Reynolds number, using the operator product expansion (OPE), Polyakov reduces the problem of computation of correlation functions in the inertial subrange, to the solution of a certain partial differential equation [22,23] (see also Ref. [28], about the generalization of Polyakov's approach) to find the probability density and scaling exponent of the moments of the "longitudinal" velocity difference in the three-dimensional strong turbulence.

In this paper we consider three-dimensional isotropic turbulence without pressure, which is described by the Navier–Stokes equations, when the pressure gradient is negligible. We derive Polyakov's master equations in higher dimensions and solve them, in the three dimensions. We derive the exact exponent of two-point density correlation functions and the energy spectrum exponent. We also discuss the time scale for which our exact results are valid for strong 3D turbulence in the presence of the pressure.

2. Turbulence without pressure in three dimensions

We consider the following quantity,

$$e_{\lambda} = \rho(x,t)\exp[\boldsymbol{\lambda}\cdot\boldsymbol{u}(x)], \tag{1}$$

where ρ and $\boldsymbol{u}$ are the density and the velocity satisfying the Navier–Stokes equations,

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \nu\nabla^2\boldsymbol{u} - \frac{\boldsymbol{\nabla}p}{\rho} + \boldsymbol{f}(\boldsymbol{x},t), \tag{2}$$

$$\rho_t + \partial_\alpha(\rho u_\alpha) = 0, \tag{3}$$

where p and ν are the pressure and viscosity, respectively. The stirring force $\boldsymbol{f}(\boldsymbol{x},t)$ is a Gaussian random force with the following correlation,

$$\langle f_\mu(\boldsymbol{x},t) f_\nu(\boldsymbol{x}',t')\rangle = k_{\mu\nu}(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'), \tag{4}$$

where $\mu,\nu = x_1, x_2, \ldots, x_N$.

We start with the situation when $\boldsymbol{\nabla}p \simeq 0$. This mode has a characteristic time of $T_p \simeq \Delta L/C_s$, where ΔL and C_s are the typical dimension of the inertial range of the system and the velocity of sound in the turbulent system, respectively. The existence of this time scale means that for the times $t < T_p$, we can ignore the pressure term in the Navier–Stokes equations and therefore our results are valid only for this time scale.

To investigate the statistical description of Eqs. (2) and (3), we consider the following two-point generating functional,

$$F_2(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{x}_1,\boldsymbol{x}_2) = \langle\rho(\boldsymbol{x}_1)\rho(\boldsymbol{x}_2)\exp[\boldsymbol{\lambda}_1\cdot\boldsymbol{u}(\boldsymbol{x}_1)+\boldsymbol{\lambda}_2\cdot\boldsymbol{u}(\boldsymbol{x}_2)]\rangle. \tag{5}$$

We write Eqs. (2) and (3) in two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ for $u_1, u_2, \ldots, u_N$ and $\rho(x)$ and multiply the equations in $\rho(\boldsymbol{x}_2)$, $\lambda_{1x_1}\rho(\boldsymbol{x}_1)\rho(\boldsymbol{x}_2), \ldots, \lambda_{1x_N}\rho(\boldsymbol{x}_1)\rho(\boldsymbol{x}_2)$ and $\rho(\boldsymbol{x}_1)$, $\lambda_{2x_1}\rho(\boldsymbol{x}_1)\rho(\boldsymbol{x}_2)\ldots$, and $\lambda_{2x_N}\rho(\boldsymbol{x}_1)\rho(\boldsymbol{x}_2)$, respectively.

We add the equations and multiply the result to $\exp[\boldsymbol{\lambda}_1\cdot\boldsymbol{u}(\boldsymbol{x}_1)+\boldsymbol{\lambda}_2\cdot\boldsymbol{u}(\boldsymbol{x}_2)]$, and averaging with respect to external random force, we find

$$\partial_t F_2 + \sum_{\{i=1,2\}\mu=x_1,\ldots,x_N}\frac{\partial}{\partial\lambda_{i,\mu}}\partial_{\mu_i}F_2 = \sum_{\{i=1,2\},\mu=x_1,\ldots,x_N} C_{i,\mu} + D_2, \tag{6}$$

where $C_{i,\mu}$ and D_2 are

$$\sum_{\{i=1,2\},\mu=x_1,\ldots,x_N} C_{i,\mu} = \sum_{\{i,j=1,2\},\mu,\nu=x_1,\ldots,x_N}\lambda_{i,\mu}\lambda_{j,\nu}k_{\mu\nu}(\boldsymbol{x}_i-\boldsymbol{x}_j)F_2 \tag{7}$$

and

$$D_2 = \langle\nu\rho(\boldsymbol{x}_1)\rho(\boldsymbol{x}_2)[\boldsymbol{\lambda}_1\cdot\nabla^2\boldsymbol{u}(\boldsymbol{x}_1)+\boldsymbol{\lambda}_2\cdot\nabla^2\boldsymbol{u}(\boldsymbol{x}_2)] \times\exp[\boldsymbol{\lambda}_1\cdot\boldsymbol{u}(\boldsymbol{x}_1)+\boldsymbol{\lambda}_2\cdot\boldsymbol{u}(\boldsymbol{x}_2)]\rangle, \tag{8}$$

where we have used Novikov's theorem. D_2 is known as the anomaly term [5].

Now we consider the anomaly term in the limit of small ν or high Reynolds numbers. It is noted that this term can not be written in terms of F_2. To find its structure we consider the symmetries of the basic equations. The basic equations are Galilean invariant and are also invariant under the rescaling of density as

$\rho \to \alpha\rho$. On the other hand, the final expression for D_2 must contain the ultraviolet finite operators $\nabla \boldsymbol{u}$, ρ and $e^{\boldsymbol{\lambda}\cdot\boldsymbol{u}}$. The only finite combination satisfying the rescaling $\rho \to \alpha\rho$ is $\rho e^{\boldsymbol{\lambda}\cdot\boldsymbol{u}}$ (see Ref. [24] for more detail). Therefore, D_2 has the following form,

$$D_2 = aF_2, \tag{9}$$

where a is generally a function of λ_1 and λ_2.

As mentioned above, we have to calculate the transient solution of Eq. (6). We find the temporal part of F_2 as e^{-t/T_p} and therefore we write $F_2 = e^{-t/T_p}\tilde{F}_2$; by substituting it into Eq. (6), one can find the following equation for $\tilde{F}_2$,

$$\sum_{\{i=1,2\},\mu=x_1,\dots,x_N} \frac{\partial}{\partial\lambda_{i,\mu}}\partial_{\mu_i}\tilde{F}_2 - \sum_{\{i,j=1,2\},\mu,\nu=x_1,\dots,x_N} \lambda_{i,\mu}\lambda_{j,\nu}k_{\mu\nu}(\boldsymbol{x}_i - \boldsymbol{x}_j)F_2 = a'\tilde{F}_2, \tag{10}$$

where $a' = a + 1/T_p$.

Also in this Letter, we suppose that $k_{\mu\nu}$ has the following form,

$$k_{\mu\nu}(\boldsymbol{x}_i - \boldsymbol{x}_j) = k(0)\left(1 - \frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|^2}{2L^2}\delta_{\mu,\nu} - \frac{(\boldsymbol{x}_i - \boldsymbol{x}_j)_\mu(\boldsymbol{x}_i - \boldsymbol{x}_j)_\nu}{L^2}\right) \tag{11}$$

with $k(0), L = 1$.

Now let us consider the Eq. (10) in three dimensions. We change the variables as, $\boldsymbol{x}_\pm = \boldsymbol{x}_1 \pm \boldsymbol{x}_2$, $\boldsymbol{\lambda}_+ = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_- = \frac{1}{2}(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)$ and consider the spherical coordinates, so that $x_- : (r,\theta,\varphi)$ and $\lambda_- : (\rho',\theta',\varphi')$. Direct calculation shows that

$$\begin{aligned}
\sum_{i=1,\mu=x,y,z}^{3} \frac{\partial}{\partial\lambda_{i,\mu}}\partial_{\mu_i} &= \sum_{\mu=x,y,z} \frac{\partial}{\partial\lambda_{-\mu}}\partial_{\mu_-} = \cos\gamma\,\partial_r\partial'_\rho \\
&+ \frac{\sin\theta\cos\theta'\cos(\varphi-\varphi') - \cos\theta\sin\theta'}{\rho'}\partial_r\partial_{\theta'} \\
&+ \frac{\sin\theta\sin(\varphi-\varphi')}{\rho'\sin\theta'}\partial_r\partial_{\varphi'} \\
&+ \frac{\sin\theta'\cos\theta\cos(\varphi-\varphi') - \cos\theta'\sin\theta}{r}\partial'_\rho\partial_\theta \\
&+ \frac{\cos\theta\sin(\varphi-\varphi')}{r\rho'\sin\theta'}\partial_\theta\partial_{\varphi'} \\
&- \frac{\cos\theta'\sin(\varphi-\varphi')}{r\rho'\sin\theta}\partial_{\theta'}\partial_\varphi + \frac{\cos(\varphi-\varphi')}{r\rho'\sin\theta\sin\theta'}\partial_\varphi\partial_{\varphi'} \\
&+ \frac{\cos\theta\cos\theta'\cos(\varphi-\varphi') + \sin\theta\sin\theta'}{r\rho'}\partial_\theta\partial_{\theta'} \\
&- \frac{\sin\theta'\sin(\varphi-\varphi')}{r\sin\theta}\partial'_\rho\partial_\varphi
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\sum_{\{i,j=1,2\},\mu,\nu=x,y,z} \lambda_{i,\mu}\lambda_{j,\nu}k_{\mu\nu}(\boldsymbol{x}_i - \boldsymbol{x}_j) \\
&= [r^2\rho'^2 + 2(x_-\lambda_{-x} + y_-\lambda_{-y} + z_-\lambda_{-z})^2] \\
&= r^2\rho'^2(1 + 2\cos^2\gamma),
\end{aligned} \tag{13}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$.

Now using Eqs. (12), (13) for isotropic turbulence we obtain

$$\begin{aligned}
&\left(s\partial_r\partial'_\rho - \frac{s(1-s^2)}{r\rho'}\partial_s^2 + \frac{1+s^2}{r\rho'}\partial_s + \frac{1-s^2}{\rho'}\partial_r\partial_s + \frac{1-s^2}{r}\partial'_\rho\partial_s - r^2\rho'^2(1+2s^2)\right)\tilde{F}_2 \\
&= a'(\rho')\tilde{F}_2,
\end{aligned} \tag{14}$$

where $\cos\gamma = s$. The ρ' dependence of the $a'(\rho')$ anomaly must be chosen to conform to the scaling and can be different depending on the scaling properties of the force correlation functions. In general, in the case of isotropic turbulence, the stirring correlation function behaves as $k_{\mu,\nu} \sim 1 - r^\eta$, where in our case we have $\eta = 2$. Therefore, a' must depend on ρ' as follows, $a'(\rho') = a'_0\rho'^\sigma$, where $\sigma = (2-\eta)/(1+\eta)$. It is evident that for our case a' is independent of ρ'. Let us consider the universal scaling invariant solution of Eq. (33) in the following form,

$$\tilde{F}_2(\rho', r) = g(r)F(\rho' r^\delta) \quad g(r) = r^{-\alpha_3} \tag{15}$$

where $\delta = \frac{1}{3}(\eta + 1)$, and α_3 is the exponent of two-point correlation functions of density and also using Eq. (11) we find $\delta = 1$.

We substitute Eq. (15) into Eq. (14), and find the following relation for $F(\rho' r)$,

$$\Big(- \frac{\alpha_3 s}{r}\partial'_\rho + s\partial'_\rho\partial_r - \frac{s(1-s^2)}{r\rho'}\partial_s^2 + \frac{1+s^2}{r\rho'}\partial_s - \alpha_3\frac{1-s^2}{r\rho'}\partial_s + \frac{1-s^2}{\rho'}\partial_s\partial_r + \frac{1-s^2}{r}\partial_s\partial'_\rho - r^2\rho'^2(1+2s^2)\Big)F(\rho' r) = a'_0 F(\rho' r), \qquad (16)$$

where $a'_0 = a_0 + 1/T_p = \text{const}$.

Since the two-point density correlators exist, in the limit of $\rho' \to 0$, $F(\rho' r)$ tends to a constant and thus we have to look for the solution of F among the family of positive, finite and normalizable solutions of Eq. (16). On the other hand, taking the Laplace transformation of the above equation, one can show that, to consider the physical solution, so that $\langle \boldsymbol{u}(\boldsymbol{x}_1) - \boldsymbol{u}(\boldsymbol{x}_2)\rangle = 0$, we have to consider the case $a'_0 = 0$ or $a_0 = -1/T_p$. However, for different types of correlation for the stirring force, e.g. $k_{\mu,\nu} \sim 1 - r^\eta$, with $\eta \neq 2$, we have to include a'_0 [23].

Now, we propose the following ansatz for $F(\rho' r)$, with $z = \rho' r$,

$$F(z,s) = e^{z^\gamma f(s)}. \qquad (17)$$

Using Eq. (16), we find $\gamma = 3/2$ and $f(s)$ satisfy the following equations,

$$\tfrac{9}{4} s f(s)^2 + 3 f(s) f'(s)(1-s^2) + f'(s)^2(-s+s^3) = (1+2s^2),$$

$$-s(1-s^2) f(s)'' + [(4+\alpha_3) - (2+\alpha_3)s^2] f'(s) + (\tfrac{9}{4} + \tfrac{3}{2}\alpha_3) s f(s) = 0 \qquad (18a)$$

also from Eq. (18a), one can derive the following initial conditions for $f(s)$,

$$f(1) = \frac{2}{\sqrt{3}}, \quad f'(1) = \frac{\sqrt{3}+\sqrt{11}}{4}. \qquad (19)$$

It is interesting to note that the equation for $f(s)$ (i.e. Eq. (18a)) is the same as equation which is found in the instanton approach [8].

The function $f(s)$ has the following expansion around $s = 1$,

$$f(s) = \frac{2}{\sqrt{3}} + \frac{\sqrt{3}+\sqrt{11}}{4}(s-1) + \frac{5\sqrt{33}-61}{32(3\sqrt{3}-2\sqrt{11})}(s-1)^2 + \dots \qquad (20)$$

Now using the boundary conditions on $f(s)$ (i.e. Eq. (19)), and positivity of the probability distribution function we find

$$\alpha_3 = \frac{12+\sqrt{33}}{6} \simeq 2.9574. \qquad (21)$$

Noting the fact that ρ' has dimension -1, we can find the following scaling relation for the density of the energy $\epsilon(x)$,

$$\epsilon(\alpha x) = \alpha^{\Delta}\epsilon(x), \qquad (22)$$

where $\Delta = 1 - \frac{1}{12}\sqrt{33}$, and therefore we can determine the behaviour of the energy spectrum exactly as

$$E(k) \sim k^{-\beta}, \quad \beta = 2 - \frac{\sqrt{33}}{12} \simeq 1.52128. \qquad (23)$$

This behaviour of the energy spectrum is known as the non-Kolmogorov power law which has been observed experimentally [25,27] and also in numerical simulations [25,26].

Numerical calculations have been performed in Refs. [25,26], where was used the Wiener–Hermit expansion. It was shown that the energy spectrum behaves as $E(k) \sim k^{-1.521}$ for systems without boundaries (i.e. free turbulence) and also for a finite system this spectrum is not stable. In Refs. [25,26] it has been shown that in the inertial subrange for a finite system the energy spectrum starts with a slope -1.521, and after a moderate time which is less than the characteristic time $T_c \simeq L/u_{\rm rms}$ (where L and $u_{\rm rms}$ are the large scale of the system and the rms value of the initial velocity fluctuation, respectively) the equilibrium is attained and has transformed to $-5/3$. In other words, the $-5/3$ law is the stable algebraic spectrum for the Navier–Stokes equations after a time of order T_c.

The experimental results (reported by Wissler [25,27]) show that the non-Kolmogorov spectrum has been observed also experimentally only for moderate times less than T_c. It is noted that in general for a turbulent flow $u_{\rm rms} \leqslant C_s$ and therefore the pressure time scale has the property that $T_p \leqslant T_c$. Also an energy spectrum with smaller exponents than the Komogorov ones has been observed in real-life experiments when the Reynolds number is not sufficiently large enough. The value of a_0 (which is proportional to the inverse of the Reynolds number) shows that one can interpret our short-time asymptotic also with these experiments[1]. In this paper the question of under what condition an equilibrium is attained (after the moderate time) remains open, though judged by experiments and numerical calculations.

Finally we can derive the PDF for the velocity difference and show that it tails as $e^{-\alpha u^3}$ in the limit $|u| \to +\infty$, which is in agreement with other approaches [18] for three-dimensional turbulence.

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References

[1] A. Polyakov, Nucl. Phys. B 396 (1993) 367.
[2] A. Migdal, Int. J. Mod. Phys. A 9 (1994) 1197.
[3] V. L'vov, I. Procaccia, chao-dyn/9502010.
[4] J. Bouchaud, M. Mezard, G. Parisi, Phys. Rev. E 52 (1995) 3656.
[5] A. Polyakov, Phys. Rev. E 52 (1995) 6183.
[6] E. Balkovsky, G. Falkovich, I. Kolokolov, V. Lebedev, JETP Lett. 61 (1995) 1012.
[7] M. Chertkov, Phys. Rev. E 55 (1997) 2722, chao-dyn/9606011.
[8] V. Gurarie, A. Migdal, Phys. Rev. E 54 (1996) 4908.
[9] J.P. Bouchaud, M. Mezard, cond-mat/9607006.
[10] K. Gawedzki, chao-dyn/9610003.
[11] D. Antonov, hep-th/9612005.
[12] P. Tomassini, chao-dyn/9706013.
[13] A.N. Kolmogorov, C.R. Acad. Sic. USSR 30 (1941) 301.
[14] M. Kardar, G. Parisi, Y. Zhang, Phys. Rev. Lett. 56, (1986) 889.
[15] A.S. Monin, A.M. Yaglom, Statistical Fluid Mechanics (MIT Press, Cambrige, MA, 1975).
[16] C. DeDominicis, P.C. Martin, Phys. Rev. A 19 (1979) 419.
[17] V. Yakhot, S.A. Orszag, J. Sci. Comp. 1 (1986) 3.
[18] G. Falkovich, V. Lebedev, chao-dyn/9708002.
[19] A. Noullez, G. Wallaec, W. Lempert, R.B. Miles, U. Frisch, J. Fluid Mech. 339 (1997) 287.
[20] R.H. Kraichnan, Y. Kimura, in: Progress in Turbulence Research (1994) p. 19.
[21] J.M. Burgers, The Nonliner Diffusion Equation (Reidel, Boston, 1974).
[22] M.R. Rahimi Tabar, S. Rouhani, B. Davoudi, Phy. Lett. A 212 (1996) 60.
[23] S. Boldyrev, Phys. Rev. E 55 (1997) 6907, hep-th/9610080.
[24] S. Boldyrev, hep-th/9707255.
[25] T. Funaki, W.A. Woyczynski, Nonlinear Stochastic PDEs (Springer, 1995) pp. 293–312.
[26] T.C. Chung, PhD Thesis, University of California at Los Angeles (1991).
[27] J.B. Wissler, presentation to meeting of the Div. of Fl. Dynamics, Am. Phys. Soc. Nov. (1993).
[28] V. Yakhot, Princeton Univ. Preprint, chao-dyn/9708016.

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