



ELSEVIER

9 July 1998

PHYSICS LETTERS B

Physics Letters B 431 (1998) 85–89

Zamalodchikov's c -theorem and the logarithmic conformal field theory

M.R. Rahimi-Tabar^{b,c}, S. Rouhani^{a,b}^a Sharif University of Technology, P.O. Box 11365-9161, Tehran, Iran^b Dept. of Physics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran^c Institute for Studies in Theoretical Physics and Mathematics, P.O. Box: 19395-5746, Tehran, Iran

Received 10 February 1998

Editor: L. Alvarez-Gaumé

Abstract

We consider perturbation of a conformal field theory by a pair of relevant logarithmic operators and calculate the beta function up to two loops. We observe that the beta function can not be derived from a potential. Thus the renormalization group trajectories are not always along decreasing values of the central charge. However there exists a domain of structure constants in which the c -theorem still holds. © 1998 Elsevier Science B.V. All rights reserved.

PACS: 11.25.Hf; 11.25.Db; 11.10.Gh

Keywords: Conformal field theory; c -theorem; Renormalization

1. Introduction

An important theorem was put forward by Zamalodchikov regarding the perturbation of conformally invariant field theories (CFT), which is known as c -theorem [1]. The c -theorem states that there exists a function C , of the coupling constants which is non-increasing along the trajectories of the renormalization group and its stationary points coincide with the fixed points of the renormalization group. At these fixed points C takes the value of the central charge.

On the other hand it is interesting to investigate the validity of the Zamalodchikov's theorem beyond its original domain. One such group of theories are the logarithmic conformal field theories (LCFT). It has been shown by Gurarie [2] that CFTs exist in

which at least two primary fields have equal conformal dimensions. Such a pair then have logarithms in their correlation functions.

The Logarithmic fields (operators) in CFT were first studied by Gurarie in the $c = -2$ model [2]. After Gurarie, these logarithms have been found in a multitude of other models such as the WZNW-model on the $GL(1,1)$ [3], the gravitationally dressed CFTs [4], $c_{p,1}$ and non-minimal $c_{p,q}$ models [2,5–7], critical disordered models [8,9,35], and the WZNW models at level 0 [10,11]. They play a role in the study of critical polymers and percolation [5,6,12,13], 2D-turbulence [14–18,36] and quantum Hall states [19–21]. They are also important for studying the problem of recoil in the string theory and D-branes [10,22–25], as well as target space symmetries in string theory [10]. The representation theory of the

Virasoro algebra for LCFT was developed in [26]. The origin of the LCFT has been discussed in [27,28,34].

Perturbing a stable fixed point by logarithmic operators has many repercussions. Firstly logarithmic operators come in at least pairs of relevant operators, thus one always has to deal with a system of equations in renormalization flow trajectories. Secondly the logarithmic response changes the renormalization flow. Thirdly non-unitarity causes negative norms and this may affect the c -theorem.

Unitarity is a prerequisite of the c -theorem and one may expect a break down of the c -theorem for non-unitary theories. Although most realizations of LCFT's so far have been non-unitary, but unitary LCFT's may also exist [11]. Therefore c -theorem within the context of LCFT's is interesting from two different points of view; it may hold under certain conditions even for non-unitary LCFT's, it restricts the unitary LCFT's.

2. The c -theorem

The proof of the c -theorem is based on conservation of the energy-momentum tensor and positivity, here we follow the proof given by Cardy [29]. In two dimensions the energy-momentum tensor has three independent components,

$$T = T_{zz}, \quad \bar{T} = T_{\bar{z}\bar{z}}, \quad U = T_{z\bar{z}} \quad (1)$$

At a fixed point the theory is conformally invariant, the beta function vanishes and $U = 0$. Thus T depends solely on z and \bar{T} on \bar{z} . The conservation of the energy-momentum tensor results in:

$$\partial_{\bar{z}} T + \frac{1}{4} \partial_z U = 0 \quad \partial_z \bar{T} + \frac{1}{4} \partial_{\bar{z}} U = 0 \quad (2)$$

We are concerned with the perturbation of a fixed point Hamiltonian by an operator Φ :

$$\mathcal{H} = H_* + g\Phi \quad (3)$$

The renormalization flow of the coupling constant g is then given by calculating the change in the correlation functions of the theory perturbatively:

$$\langle \cdots \rangle = \langle \cdots \rangle_* + g \int \langle \Phi \cdots \rangle \quad (4)$$

We can now use the operator product expansion on the rhs. of Eq. (4):

$$T(z)\Phi(z_1) = \frac{h}{(z-z_1)^2} \Phi(z) + \frac{1-h}{z-z_1} \partial\Phi(z) \quad (5)$$

We observe that the rhs of Eq. (4) is divergent thus needs regularization. Consequently $\partial_{\bar{z}} T$ no longer vanishes and we find:

$$\partial_{\bar{z}} T = -\pi(1-h)\partial_z \Phi \quad (6)$$

then the conservation of the energy-momentum tensor implies that:

$$U = -4\pi g(1-h)\Phi \quad (7)$$

Here U is the response of the action to the scale transformation $z \rightarrow \lambda z$. This is valid all the way along a trajectory leaving a fixed point until another fixed point is reached. At the second point the UV behaviour changes. Let us consider the flow from a UV fixed point, to a relatively IR fixed point. Given the spin structure of the three components of energy-momentum tensor, the following holds:

$$\begin{aligned} \langle T(z)T(0) \rangle &= \frac{F(z\bar{z})}{z^4}, & \langle \bar{T}(z)\bar{T}(0) \rangle &= \frac{G(z\bar{z})}{z^4}, \\ \langle U(z)U(0) \rangle &= \frac{H(z\bar{z})}{z^2\bar{z}^2} \end{aligned} \quad (8)$$

using the conservation of the energy-momentum tensor (i.e. Eq. (2)), we have

$$\dot{F} + \frac{1}{4}(\dot{G} - 3G) = 0, \quad \dot{G} - G + \frac{1}{4}(\dot{H} - 2H) = 0 \quad (9)$$

where $\dot{F} = z\bar{z}F(z\bar{z})$. Defining $C = 2F - G - \frac{3}{8}H$, we have

$$\dot{C} = -\frac{3}{4}H \quad (10)$$

Now in unitary theories we have $H > 0$, thus C is a non-decreasing function, and it is stationary only when U is zero, that is at the conformally invariant points. Furthermore the quantities G and H vanish at the fixed point and $F = c/2$, thus we have $C = c$. This proof can be easily extended to the case of more than one operator.

3. The logarithmic conformal field theories

In its simplest version a logarithmic field theory is characterized by a pair of fields which mix due to a scale transformation:

$$\Phi(z) \rightarrow \lambda^{-x} \Phi(z) \quad (11)$$

$$\Psi(z) \rightarrow \lambda^{-x} (\Psi(z) - \log(\lambda) \Phi) \quad (12)$$

Note that formally one can think of Ψ as derivative of Φ with respect to x [28,30,34]. The OPE with the energy momentum tensor likewise changes:

$$T(z) \Phi(z_1) = \frac{h\Phi}{(z-z_1)^2} + \frac{1-h}{(z-z_1)} \partial\Phi \quad (13)$$

$$T(z) \Psi(z_1) = \frac{h\Psi}{(z-z_1)^2} + \frac{\Phi}{(z-z_1)^2} + \frac{1-h}{(z-z_1)} \partial\Psi - \frac{1}{(z-z_1)} \partial\Phi \quad (14)$$

The consistency of invariance under the action of Virasoro algebra generators requires the two-point functions of Ψ and Φ to have an unusual form:

$$\langle \Phi(z) \Phi(0) \rangle = 0 \quad (15)$$

$$\langle \Phi(z) \Psi(0) \rangle = bz^{-2x} \quad (16)$$

$$\langle \Psi(z) \Psi(0) \rangle = z^{-2x} (d - 2b \log(z)) \quad (17)$$

where b and d are constants. It is this above property which has an important bearing on the c -theorem. We perturb the fixed point Hamiltonian \mathcal{H}_* by a pair of operators Ψ and Φ using two coupling constants g_1 and g_2 :

$$\mathcal{H} = H_* + \int d^2z (g_1 \Phi + g_2 \Psi) \quad (18)$$

To make the coupling constants dimensionless, and also maintain the invariance given by Eqs. (11,12), we rewrite the above expression as follows:

$$\mathcal{H} = H_* + \int d^2z (g_2 a^{(x-2)} \Psi + G_1 a^{(x-2)} \Phi) \quad (19)$$

where a is the lattice constant and the coupling constant G_1 is:

$$G_1 = g_1 + \log(a) g_2 \quad (20)$$

The function $C(g_1, g_2)$ now can be calculated using the two point functions and Eq. (7) as:

$$C(g_1, g_2) = c^* - 6\pi^2 (2g_1 g_2 (2-x)b - g_2^2 b + g_2^2 (2-x)d) \quad (21)$$

Clearly C does not always decrease as result of a change in scale, but it does if the following condition holds:

$$2(2-x)g_1 > (b-d(2-x))g_2 \quad (22)$$

This in turn holds if $x \langle 2, g_1 \rangle > 0, g_2 > 0$ and:

$$b = d(2-x) \quad (23)$$

To calculate the renormalization flow we need the OPE coefficients. Up to two loops, the result of ordinary CFT still holds [29] provided we replace g_1 with G_1 :

$$\dot{g}_i = (2-x_i)g_i - \pi c_i^{jk} g_k g_j \quad (24)$$

Note that in LCFT the structure functions depend on $\log(r)$ through logarithmic terms and one has to take care when applying the above equation. To proceed further we need the OPE of the fields Φ and Ψ [28,34]:

$$\Phi(z) \Phi(0) = \dots + z^{-x} (A - B \log(z)) \Phi(0) + Bz^{-x} \Psi(0) \quad (25)$$

$$\Phi(z) \Psi(0) = \dots + bz^{-2x} + z^{-x} (D - (A-E) \log(z) - B(\log(z))^2) \Phi(0) + z^{-x} (E - B \log(z)) \Psi(0) \quad (26)$$

$$\Psi(z) \Psi(0) = \dots + (d - 2b \log(z)) z^{-2x} + z^{-x} (G - (2D - K) \log(z) + (A - 2E) (\log(z))^2 + B \log(z)^3) \Phi(0) + z^{-x} (K - 2E \log(z) + B \log(z)^2) \Psi(0) \quad (27)$$

Here we have assumed that the 1-point functions of fields with zero conformal dimension vanishes, but

this is need not be the case in nonunitary theories. The operator product expansion may change if this assumption is removed, and thus the renormalization flow will change. we shall deal with this case in a future work.

After some algebra we obtain the renormalization flows:

$$\dot{g}_1 = (2-x)g_1 - g_2 - \pi A g_1^2 - \pi G g_2^2 - 2\pi D g_1 g_2 + \dots \quad (28)$$

$$\dot{g}_2 = (2-x)g_2 - \pi K g_2^2 - 2\pi E g_1 g_2 - \pi B g_1^2 + \dots \quad (29)$$

These equations clearly do not admit a potential, and even at the one loop approximation they have a Jordan form and cannot be diagonalized.

However the UV fixed point ($g_1 = g_2 = 0$) is a stable point depending on whether $2-x$ is positive or not, in non-unitary theories, x is negative thus the UV point is always unstable. Although a potential does not exist it may still be that the flow minimizes some function such as C as defined above. However a one loop calculation of C has indicated that C is not always decreasing.

4. Discussion

The above results can easily be generalized to the case where the Jordan cell has more than two members. If the Hamiltonian is perturbed by more than two logarithmic fields:

$$\mathcal{H} = H_* + \sum_{\alpha=1}^N \Phi_{\alpha} \quad (30)$$

we then derive the following 1-loop equations:

$$\dot{g}_N = (2-x)g_N \quad (31)$$

$$\dot{g}_{N-1} = (2-x)g_{N-1} - g_N \quad (32)$$

and

$$\dot{g}_1 = (2-x)g_1 - (N-1)g_2 \quad (33)$$

To summarize we observe that the c -theorem does not always hold in logarithmic conformal field theories, but under certain conditions it may hold. When dealing with non-unitary theories this is not a

disaster, but if we find unitary LCFT's this result will put a restriction on their structure constants. Some authors have discussed the validity of the c -theorem in a wider context [5,31–33] it may be that a different definition of C , is necessary to cover cases such as disorder. We also observe that LCFT's can be formulated in terms of nilpotent parameters [34], we suspect that the above analysis should have a transparent form if expressed in this terminology, work in this direction is under progress.

References

- [1] A.B. Zamalodchikov, JETP Lett. 43 (1986) 731.
- [2] V. Gurarie, Nucl. Phys. B 410 [FS] (1993) 535.
- [3] L. Rozansky, H. Saleur, Nucl. Phys. B 376 (1992) 461.
- [4] A. Bilal, I.I. Kogan, hep-th/9407151; Nucl. Phys. B 449 (1995) 569.
- [5] M.A. Flohr, Int. J. Mod. Phys. A 11 (1996) 4147; Int. J. Mod. Phys. A 12 (1997) 1943.
- [6] H.G. Kausch, hep-th/9510149.
- [7] M.R. Gaberdiel, H.G. Kausch, Nucl. Phys. B 447 (1996) 293; Phys. Lett. B 386 (1996) 131.
- [8] J.S. Caux, I.I. Kogan, A.M. Tsvetik, Nucl. Phys. B 466 (1996) 444.
- [9] Z. Maassarani, D. Serban, Nucl. Phys. B 489 (1997) 603.
- [10] I.I. Kogan, N.E. Mavromatos, Phys. Lett. B 375 (1996) 111.
- [11] J.S. Caux, I. Kogan, A. Lewis, A.M. Tsvetik, Nucl. Phys. B 289 (1997) 469, hep-th/9606138.
- [12] H. Saleur, Nucl. Phys. B 382 (1992) 486.
- [13] G.M. Watts, cond-mat/9603167.
- [14] M.R. Rahimi Tabar, S. Rouhani, Annals of Phys. 246 (1996) 446.
- [15] M.R. Rahimi Tabar, S. Rouhani, Nuovo Cimento B 112 (1997) 1079.
- [16] M.R. Rahimi Tabar, S. Rouhani, Europhys. Lett. 37 (1997) 447.
- [17] M.R. Rahimi Tabar, S. Rouhani, Phys. Lett. A 224 (1997) 331.
- [18] M.A.I. Flohr, Nucl. Phys. B 482 (1996) 567.
- [19] M.A. Flohr, Mod. Phys. Lett. A 11 (1996) 55.
- [20] X.G. Wen, Y.S. Wu, Y. Hatsugai, Nucl. Phys. B 422 [FS] (1994) 476.
- [21] V. Gurarie, M.A.I. Flohr, C. Nayak, The Haldane-Rezai Quantum Hall states and Conformal Field Theory, cond-mat/9701212.
- [22] I.I. Kogan, N.E. Mavromatos, J.F. Wheeler, Phys. Lett. B 387 (1996) 483.
- [23] V. Periwal, O. Tafjord, Phys. Rev. D 54 (1996) 4690.
- [24] J. Ellis, N.E. Mavromatos, D.V. Nanopoulos, Int. J. Mod. Phys. A 12 (1997) 2639, hep-th/9605046.
- [25] D. Bernstein, R. Corrado, W. Fishler, S. Paban, M. Rosali, Phys. Lett. B 384 (1996) 93.

- [26] F. Rohsiepe, hep-th/9611160.
- [27] I.I. Kogan, A. Lewis, hep-th/9705240.
- [28] M.R. Rahimi Tabar, A. Aghamohammadi, M. Khorrami, Nucl. Phys. B 497 (1997) 555.
- [29] J. Cardy, in: Brezin, J. Zinn-Justin (Eds.), Fields, Strings and Critical Phenomena, Les Houches Lectures, 1989.
- [30] A. Shafiekhani, M.R. Rahimi Tabar, Int. J. Mod. Phys. A 12 (1997) 3723.
- [31] D. Kurtasov, Mod. Phys. Lett. A 7 (1992) 2943.
- [32] J. Cardy, J. Phys. A 29 (1996) 1897.
- [33] P. Damgaard, J. Lacki, Partition Function Zeros of an Ising Spin Glass, cond-mat/9407119.
- [34] M.A.I. Flohr, Singular Vectors in Logarithmic Conformal Field Theory, hep-th/9707090.
- [35] J.S. Caux, N. Taniguchi, A.M. Tselik, cond-mat/9801055.
- [36] S. Skoulakis, S. Thomas, cond-mat/9802040.