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A Logarithmic Conformal Field Theory Solution For Two Dimensional Magnetohydrodynamics in Presence of The Alf'ven Effect

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Abstract

When Alf'ven effect is peresent in magnetohydrodynamics one is naturally lead to consider conformal field theories, which have logarithmic terms in their correlation functions. We discuss the implications of such logarithmic terms and find a unique conformal field theory with centeral charge $c = -\frac{209}{7}$, within the border of the minimal series, which satisfies all the constraints. The energy espectrum is found to be

$E(k) \sim k^{-\frac{13}{7}} \log k.$

1 - Introduction

There has been some work on modelling turbulence in two dimensional fluids by conformal field theory (CFT) [1-3]. Ferretti et al. [4] have generalized Polyakov's method [1] to the case of two dimensional magnetohydrodynamics (2D - MHD). We have argued that the existence of a critical dynamical index is equivalent to the Alf'ven effect [5] i.e. the equipartition of energy between velocity and magnetic modes [6]. The Alf'ven effect, reduces the number of candidate conformal field theories, but also it implies that the velocity stream function ϕ and the magnetic flux function ψ should have similar scaling dimensions. This naturally leads to logarithmic conformal field theories, Gurarie [7] has argued that such theories do exist albeit the need to extend some of the definitions of CFT to include logarithmic operators [8].

In such conformal field theories, it has been shown [7] that the correlator of two fields, has a logarithmic singularity.

$$< \psi(r)\psi(r^{'}) > \sim |r - r^{'}|^{-2h_{\psi}} \log|r - r^{'}| + \dots \tag{1}$$

Examples of such theories have been studied by many authors [9-18], a particularly interesting application is in the field of disordered systems [17,18]. In this paper we shall present a CFT with central charge $c = -\frac{209}{7}$, which is a member of the series $c_{p,1}$ [8]. This is the only CFT found so far which satisfies all the constraints of 2D-MHD, including the Alf'ven effect. In this CFT logarithmic correlators appear and in particular the energy spectrum is as follows:

$$E(k) \sim k^{-\frac{13}{7}} \log k \tag{2}$$

A logarithmic dependence was observed in computer solutions by Borue[19].

This paper is organised as follows; in section two we give a very brief summary of magnetohydrodynamics and the Alf'ven effect. In section 3 we discuss the implication of the logarithmic divergence and candidate CFT models are given in section 4 .

2 - The Alf'ven effect and conformal field theory.

The incompressible two dimensional magnetohydrodynamic (2D - MHD) system has two independent dynamical variables, the velocity stream function ϕ and the magnetic flux function ψ. These obey the pair of equations [20],

$$\frac{\partial \omega}{\partial t} = -e_{\alpha\beta}\partial_\alpha \phi \partial_\beta \omega + e_{\alpha\beta}\partial_\alpha \psi \partial_\beta J + \mu \bigtriangledown^2 \omega \tag{3}$$

$$\frac{\partial \psi}{\partial t} = -e_{\alpha\beta}\partial_\alpha \phi \partial_\beta \psi + \eta J \tag{4}$$

where the vorticity $\omega = \bigtriangledown^2 \phi$ and the current $J = \bigtriangledown^2 \psi$. The two quantitiesy μ and η are the viscosity and molecular resistivity, respectively. The velocity and magnetic fields are given in terms of ϕ and ψ :

$$V_\alpha = e_{\alpha\beta}\partial_\beta \phi \tag{5}$$

$$B_\alpha = e_{\alpha\beta}\partial_\beta \psi \tag{6}$$

and $e_{\alpha\beta}$ is the totally antisymmetric tensor, with $e_{12} = 1$. Chandrasekhar [5] has shown that the Alf'ven effect or the equipartition of energy between velocity and magnetic modes requires $V_k^2 = \alpha B_k^2$, with α of order unity. In fact he finds $\alpha = 1.62647$ for 2D - MHD. We [6] have argued that the existence of a critical dynamical index for 2D - MHD, implies the Alf'ven effect and if the conformal model holds, this implies the equality of scaling dimensions of ϕ and ψ :

$$h_\phi = h_\psi \tag{7}$$

Here the criteria of Gurarie [7] are satisfied and these two fields are logarithmically correlated. According to Gurarie [7], the operator product expansion of two fields A and B, which have two fields ϕ and ψ of equal dimension in their fusion rule, has a logarithmic term:

$$A(z)B(0) = z^{h_\phi - h_A - h_B}\{\psi(0) + \ldots + \log z(\phi(0) + \ldots)\} \tag{8}$$

to see this it is sufficient to look at four point function :

$$< A(z_1)B(z_2)A(z_3)B(z_4) >\sim \frac{1}{(z_1 - z_3)^{h_A}} \frac{1}{(z_2 - z_4)^{h_B}} \frac{1}{[x(1-x)]^{h_A + h_B - h_\phi}} F(x) \tag{9}$$

Where the cross ratio x is given by :

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \tag{10}$$

In degenerate models $F(x)$ satisfies a second order linear differential equation.

The hypergeometric equation governing the correlator of two fields in whose OPE, two other fields ψ and ϕ with conformal dimension h_ψ and $h_\psi + \epsilon$ appear, admits two solutions [10]:

$${}_2F_1(a, b, c, x) \tag{11}$$

$$x^\epsilon {}_2F_1(a + \epsilon, b + \epsilon, c + 2\epsilon, x) \tag{12}$$

where a, b, and c are sums of conformal dimensions. Clearly in the limit of $\epsilon \to 0$ these two solutions coincide. Another independent solution exists, it involves logarithms and can be generated by standard methods. Therefore in which case two independent solutions can be constructed according to :

$$\sum b_n x^n + \log x \sum a_n x^n \tag{13}$$

Now consistency of equation (12) and (8) requires :

$$< A(z_1)B(z_2)\psi(z_3) >=< A(z_1)B(z_2)\phi(z_3) > \{\log \frac{(z_1 - z_2)}{(z_1 - z_3)(z_2 - z_3)} + \lambda\} \tag{14}$$

$$< \psi(z)\psi(0) >\sim \frac{1}{z^{2h_\psi}}[\log z + \lambda^{'}] \tag{15}$$

$$< \psi(z)\phi(0) >\sim \frac{1}{z^{2h_\phi}} \tag{16}$$

where λ and $\lambda^{'}$ are constants.

Note that the correlators of this theory are annihilated by the set $(L_{-1}, L_0^2, L_- L_0, L_+ L_0)$, thus we may solve a differential equation for$< \psi(z_1)\psi(z_2) >$, which leads to logarithmic singularities [18,21]. This is compatible with the findings in [7] that this type of operators together with ordinary primary operators form the basis of the Jordan cell for the operator L_0. This fact allows us to find higher-order correlation functions for the operator ψ [21].

Thus a candidate CFT has to be logarithmic due to condition imposed by eq.(7) , the fields ϕ and ψ must have negative conformal dimensions and also satisfy the cascade condition.

Consider the fusion of two fields ϕ and ψ :

$$\phi \times \psi = \chi + ... \tag{17}$$

Such that χ is the field with minimum conformal dimension, on the right hand side. Then the magnetic potential cascade implies[6]:

$$\Delta_\phi + \Delta_\chi = -2 \tag{18}$$

The satisfaction of this set of constraints is considered in section 4, but first we must reconsider the problem of infra red divergence.

3- The Infrared problem and The Energy Spectrum:

The presence of logarithmic terms requies a reconsideration of the infrared problem. The

k-representation of the correlation is;

$$< \psi(k)\psi(-k) >= |k|^{-2-2|h_\phi|}[C_1 + \log k] \tag{19}$$

which is divergent in the limit of $k \to 0$. One can set some cut-off in the k-space to remove this divergence :

$$\begin{aligned} < \psi(x)\psi(0) > &= \int_{k>\frac{1}{R}}^{\infty} k^{-2-2|h_\phi|}[C + \log k]e^{ik\cdot x}d^2k \\ &\sim R^{2|h_\phi|}(\log R + C^{'}) - x^{2|h_\phi|}(C^{'} + \log X) + \ldots) \end{aligned} \tag{20}$$

where R is the large scale of the system. It seems that it is natural to add some condensate term [1] in momentum space to cancel the infrared divergence. The energy spectrum for this type of correlation, is

$$E(k) \simeq k^{-2|h_\phi|+1}(C + \log k) \tag{21}$$

This spectrum is compatible with the results of Ref. [19] where it has been shown that, one loop correction to the energy spectrum gives a logarithmic contribution to the energy spectrum.

4- Finding a Candidate Conformal Field Theory.

A possible candidate may exist within the $c_{p,1}$ series[8,21]. The central charge for this series is $c = 13 - 6(p + p^{-1})$. This series is particular since it has $c_{eff} = 1$. These CFT's posess $3p - 1$ highest weight representations with conformal dimensions:

$$h_{p,s} = \frac{(p-s)^2 - (p-1)^2}{4p} \quad , 1 \leq s \leq 3p - 1 \tag{22}$$

of these $2(p - 1)$ have pair wise equal dimensions. Two fields ϕ_s and $\phi_{s'}$ have equal and negative weights provided that $s + s' = 2p$, $(s \neq 1, 2p - 1)$. Let us adopt such a pair as candidates for the fluxes ϕ and ψ. Then the fusion rule gives:

$$\phi_s \times \phi_{s'} = \phi_{2p-1} + \cdots + \phi_1 \tag{23}$$

where the sum is only over the odd values . The field on the right hand side of eq.(23) with the lowest dimension is a candidate for χ. We then have the dimension of χ as:

$$\Delta_\chi = \begin{cases} \frac{-k^2}{2k+1} & p = 2k+1 \\ -\frac{1}{2}(k-1) & p = 2k \end{cases} \tag{24}$$

We can now look for the candidate values of s and p such that eq.(18) is satisfied. The only solution is given by:

$$p = 7 \qquad , \Delta_\phi = \Delta_\psi = -\frac{5}{7} \qquad , \Delta_\chi = -\frac{9}{7} \tag{25}$$

Note that for this solution χ is the field with minimum conformal dimension in this CFT. Input these values into the formula for the energy spectrum to get:

$$E(k) \sim k^{-\frac{13}{7}} \log k \tag{26}$$

In summing up we observe that the imposition of the Alf'ven effect greatly narrows the choice for candidate CFT's. From an infinte number of candidates with $h_\psi \neq h_\phi$, we end up with just one candidate with $h_\psi = h_\phi$.

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