# Exact Lyapunov exponent of the harmonic magnon modes of one-dimensional Heisenberg-Mattis spin glasses

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A mapping is developed between the linearized equation of motion for the dynamics of the transverse modes at $T=0$ of the Heisenberg-Mattis model of one-dimensional (1D) spin glasses and the (discretized) random wave equation. The mapping is used to derive an exact expression for the Lyapunov exponent (LE) of the magnon modes of spin glasses and to show that it follows anomalous scaling at low magnon frequencies. In addition, through numerical simulations, the differences between the LE and the density of states of the wave equation in a discrete 1D model of randomly disordered media (those with a *finite* correlation length) and that of continuous media (with a *zero* correlation length) are demonstrated and emphasized.

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## I. INTRODUCTION

Since the pioneering work of Anderson,[1] much research has been done on the propagation and localization of waves in disordered media. Anderson showed that, due to disorder, electronic states of three-dimensional crystals can be localized in space and that there is a disorder-induced transition from the extended to the localized phase. Mott and Twose[2] conjectured that all the eigenstates of an electron in a one-dimensional (1D) disordered potential are localized in space, which was confirmed by the scaling theory of localization, as advanced by Abrahams *et al.*[3]

A very useful quantity for determining whether a state is delocalized or localized is the Lyapunov exponent (LE) $\gamma$, which is simply the inverse of the localization length $\xi$. If $\gamma > 0$ for all energies or frequencies $\omega$, then all the states are localized; that is, the wave function $\psi(r)$ decays at large distances $r$ from the domain's center as $\psi(r) \sim \exp[-\gamma(\omega) r]$. The transition between the two states—the metal-to-insulator transition—is characterized by the LE which follows the power law, $\gamma \sim |\sigma - \sigma_c|^\nu$, where $\sigma_c$ is the critical value of disorder strength $\sigma$.

In the literature, there is rigorous proof of localization, with an exponential decay of the wave function, in 1D systems with diagonal disorder.[4] Ishii[5] showed that a powerful theorem proposed by Furstenberg and co-workers,[6] concerning the limit of products of noncommuting random variables, can be utilized in predicting a nonzero LE for the 1D model. Moreover, the LE of the 1D Anderson model with the Cauchy distribution of the potentials was calculated precisely[5] and it turned out to be nonzero for all the energies. The weak-disorder expansion of the LE was also given[7,8] through the energy band and near the band edge.[9,10] In most cases, however, the LE is calculated numerically by, for example, the transfer-matrix (TM) method.

An important implication of the wave characteristics of electrons is that the localization phenomena may also occur in the propagation of the classical waves. However, unlike electron localization in strongly disordered materials, which has proven to be a very difficult problem, classical waves, such as acoustic[11] or elastic[12] waves, do not interact with one another; therefore, their propagation in strongly heterogeneous media provides an ideal tool for studying the localization phenomena. Thus, many theoretical and experimental efforts have been devoted to the problem of the propagation of classical waves. For example, a low-frequency expansion of the LE of harmonic chains with a random distribution of masses yields

$$\gamma = \frac{\sigma^2 \omega^2}{8 m_a k_s}, \tag{1}$$

which was derived by Matsuda and Ishii.[13] Here, $m_a$ is the average mass, $k_s$ is the spring constant, and $\sigma^2$ is the variance of the mass distribution (see also below).

Another important area of research over the past 30 years in understanding disordered media has been the study of spin glasses, which are disordered magnetic systems in which the atoms are frozen in random positions. Then, the magnets associated with the atoms are "frozen" in random orientations. There is a competition between the ferromagnetic and antiferromagnetic interaction, which, due to the quenched disorder, exhibits a continuous "freezing" transition to a phase with zero net magnetization. It is believed that spin glasses have only localized eigenvectors,[14] which have made it possible to explain some of the experimental data for them. Such ideas have also led to a description of the spin glass transition in terms of a transition to an extended state at the mobility edge.[15] Although, there are certain differences in the symmetries of the field-theoretical description of the two problems, it is believed that there is a deep relation between the qualitative features of spin glasses and the localization problem.

In this paper, we derive a mapping between the random wave equation and the equation of motion for the dynamics of 1D spin glasses in order to derive an exact expression for the LE of the magnon modes of the Heisenberg-Mattis (HM) spin glasses. We first summarize what is already known

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about the LE and the density of states (DOS) of the random wave equation. The LE and the DOS of the wave equation and their dependence on the frequency $\omega$ and the variance $\sigma^2$ of the disorder are then computed using the TM method and are compared with the analytical results. The comparison helps us to understand whether there are any significant differences between the LE and the DOS of the continuous model (those with *zero* correlation length) and its discrete counterpart (which represents a disordered medium with a *finite* correlation length). We then describe the mapping between the random wave equation and the equation of motion for the 1D HM spin glasses in order to derive the new exact result for the LE of the model.

## II. ONE-DIMENSIONAL RANDOM WAVE EQUATION

Consider the classical wave equation in a medium with random densities or masses,

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} - m(x)\frac{\partial^2 \psi(x,t)}{\partial t^2} = 0, \tag{2}$$

where the density is given by $m(x)=m_a+c(x)$, with $m_a$ as the average density and $c(x)$ as a Gaussian and uncorrelated random function, such that

$$\langle c(x)\rangle = 0,$$

$$\langle c(x)c(x')\rangle = \sigma^2\delta(x-x'). \tag{3}$$

We consider a wave component with angular frequency $\omega$. Taking the temporal Fourier transformation of Eq. (2) yields

$$\frac{\partial^2 \hat{\psi}(x,\omega)}{\partial x^2} + m(x)\omega^2\hat{\psi}(x,\omega) = 0, \tag{4}$$

where $\hat{\psi}(x,\omega)$ is the Fourier transform of $\psi(x,t)$. Hereafter, for simplicity, we delete the hat sign. The log derivative of $\psi(x,\omega)$, i.e., $f(x,\omega)=\psi'(x,\omega)/\psi(x,\omega)$, satisfies the following equation:

$$f' + f^2 + \omega^2 m(x) = 0. \tag{5}$$

In one dimension, the Thouless relation[16] connects the LE to the DOS of a system. However, the LE may also be derived in terms of the probability density function (PDF) of $f$, i.e., the probability $P(\zeta,x,\omega)d\zeta$ that one finds $f(x,\omega)$ in $(\zeta,\zeta+d\zeta)$, implying that $P(\zeta,x,\omega)=\langle\delta(f-\zeta)\rangle$. The advantage of the method based on the PDF is that it can be generalized to higher-dimensional systems. The LE $\gamma$ is given by[17]

$$\gamma(\omega) = \lim_{x\to\infty}\left\langle \frac{\partial}{\partial x}\ln(\psi)\right\rangle = \langle f\rangle_\infty, \tag{6}$$

where the average is taken over the PDF of $f$ at infinity, which is the solution to the following equation:

$$\left(\zeta^2 + \omega^2 m_a + \frac{1}{2}\sigma^2\omega^4\frac{\partial}{\partial\zeta}\right)p(\zeta,\omega) = p_0, \tag{7}$$

where $p(\zeta,\omega)=\lim_{x\to\infty}P(\zeta,x,\omega)$. Equation (7) is the Airy equation which, after integration, yields the following solution:

$$p(\zeta,\omega) = \frac{2p_0}{\sigma^2\omega^4}\exp\left[-\frac{2}{\sigma^2\omega^4}\left(\frac{1}{3}\zeta^3 + \omega^2 m_a\zeta\right)\right]\int_{-\infty}^{\zeta} d\eta\,\exp\left[\frac{2}{\sigma^2\omega^4}\left(\frac{1}{3}\eta^3 + \omega^2 m_a\eta\right)\right]. \tag{8}$$

Therefore, one obtains[17]

$$\gamma(\omega) = \int \zeta p(\zeta,\omega)d\zeta. \tag{9}$$

Substituting $p(\zeta,\omega)$ from Eq. (8), one obtains, after some algebra, the following *exact* relation:[17]

$$\gamma(\omega) = \frac{\omega^2}{2}\left(\frac{I_+}{I_-}\right), \tag{10}$$

where

$$I_\pm(\omega) = \int_0^\infty du\;\, u^{\pm 1/2}\exp\left(-\frac{\omega^2}{6\sigma^2}u^3 - \frac{2m_a}{\sigma^2}u\right). \tag{11}$$

In the low-frequency limit ($\omega\to 0$), the integral [Eq. (11)] can be expanded in powers of $\omega$,

$$I_\pm(\omega) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\sigma^{4n+2\pm 1}}{(2m_a)^{3n+1\pm 1/2}}\Gamma(3n+1\pm 1/2)\omega^{2n}, \tag{12}$$

which then yields[17] an exact expression for the LE in the low frequency ($\omega\to 0$),

$$\gamma(\omega) = \frac{\sigma^2\omega^2}{8m_a}, \tag{13}$$

which is identical to Eq. (1) [if one sets $k_s=1$ in Eq. (1)], as derived by Matsuda and Ishii[13] using a perturbative method. At high frequencies, a suitable expansion yields the following equation for the LE:

$$\gamma = \frac{6^{1/3}\sqrt{\pi}}{2\Gamma(1/6)}\sigma^{2/3}\omega^{4/3}, \tag{14}$$

implying that at high frequencies the localization length $\xi=1/\gamma$ scales with $\omega$ as $\xi\sim\omega^{-4/3}$, i.e., localization becomes, in some sense, stronger at such frequencies.

Sheng *et al.*[18] suggested that in the presence of a diagonal disorder and at high frequencies, $\xi$ either approaches a constant or *diverges*. Equation (14) indicates that the localization length decreases at high frequencies. As discussed below, we attribute the difference between Eq. (14) and the result of the work of Sheng *et al.*[18] at high frequencies to the presence of a nonzero correlation length in the type of disorder that they included in their model. Equation (14) is valid for a completely random (with a *zero* correlation length) disorder. Thus, in the above continuum model, there is no characteristic microscopic length scale so that the propagating waves perceive a medium with the same properties at all relevant length scales.

It can be shown[17,19,20] that the exact *integrated* DOS,

$\mathcal{N}(\omega^2)$, for disordered 1D lattices is given by

$$\mathcal{N}(\omega^2) = p_0, \tag{15}$$

where $p_0$ is given by Eqs. (7) and (8). It is then straightforward to show that $\mathcal{N}(\omega^2)$ has the following exact closed form:[17,19,20]

$$\frac{1}{\mathcal{N}(\omega^2)} = \frac{\sqrt{2\pi}}{(\omega^2\sigma)^{2/3}}\int_0^\infty \frac{dx}{\sqrt{x}}\exp\left(-\frac{1}{6}x^3 - \frac{2m_a}{\omega^{2/3}\sigma^{4/3}}x\right), \tag{16}$$

which, in fact, was first derived by Frisch and Lloyd.[20] In the limit $\sigma \to 0$, Eq. (16) simplifies to

$$\mathcal{N}(\omega^2) = \frac{2}{\pi}\sqrt{m_a\omega^2}, \tag{17}$$

which is the expected result for a 1D ordered system with a constant density.

Note that the above results pertain to random media with *nonzero* average densities. In Sec. IV, we derive the exact result for 1D media with a *zero* mean density.

## III. NUMERICAL SIMULATION: DIFFERENCE BETWEEN CONTINUOUS AND DISCRETE SYSTEMS

We begin with the discretized version of Eq. (4), which, for the wave component $\psi(x,\omega)$ at site $i$ of a 1D lattice, is given by

$$\psi_{i+1} + \psi_{i-1} - (2 - m_i\omega^2)\psi_i = 0, \tag{18}$$

where $\psi_i$ is the value of the function at site $i$. Equation (18) is then written in the TM form as

$$\begin{pmatrix}\psi_{i+1}\\ \psi_i\end{pmatrix} = \begin{pmatrix}2 - m_i\omega^2 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_i\\ \psi_{i-1}\end{pmatrix}, \tag{19}$$

with the initial normalized vector $\mathbf{v}=(1/\sqrt{2},1/\sqrt{2})^T$. To ensure the stability of the numerical method, after multiplying the transfer matrices $M$ times, we checked the length of the resulting vector, normalized it again, and then continued with the new vector.[21–24] Moreover, to ensure that the random masses were all positive, we kept $\sigma/m_a$ small and checked that the random masses were indeed positive for each frequency. The LE is then expressed in terms of vector lengths $d_\alpha$, obtained after $N$ normalization of $\mathbf{v}$, that is,

$$\gamma = \frac{1}{MN}\sum_{\alpha=1}^{N}\ln(d_\alpha). \tag{20}$$

Moreover, the error in estimating $\gamma$ is given by

$$\frac{\Delta\gamma}{\gamma} = \frac{1}{\sqrt{N}}\frac{\sqrt{\langle(\ln d_\alpha)^2\rangle - \langle\ln d_\alpha\rangle^2}}{\langle\ln d_\alpha\rangle}, \tag{21}$$

where the angular brackets indicate averaging over the sequence of $\{d_\alpha\}$. According to the Oseledec theorem,[21–25] $\gamma$ is a self-averaged quantity, and the error of its estimates approaches zero as $1/\sqrt{N}$ as $N$ increases.

Figure 1 presents the frequency dependence of the LE for the disorder standard deviations, $\sigma=0.5$, computed both analytically and numerically (using the TM method) for the average densities $m_a=0$ and 2. The numerical results indicate that for small frequencies, the LE scales with the frequencies $\gamma\sim\omega^{1.33\pm0.01}$ and $\gamma\sim\omega^{2.06\pm0.01}$ for $m_a=0$ and $m_a=2$, respectively, which is in agreement with the exact results. Moreover, Fig. 1 exhibits the effect of a nonzero average density on the scaling of the LE with the frequency. For large frequencies, we find the difference between the spectrum of the continuous random-mass model and the discrete one to be due to the presence of a high-frequency cutoff in the latter model. For small frequencies, or in the limit of long wavelengths, the propagating waves cannot be scattered by the fluctuations in the mass. The reason is that the typical length scale over which the mass fluctuates is very small in comparison with the wavelength. In the opposite limit, in which the waves' wavelength is small compared to the length scale of mass fluctuations, the medium is, indeed, in the limit of geometrical acoustics.

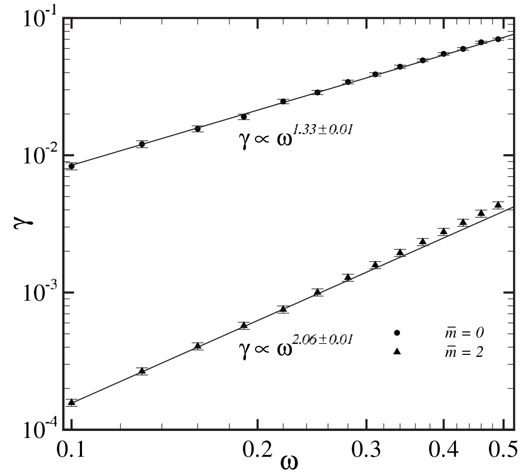


FIG. 1. Frequency dependence of the Lyapunov exponent $\gamma$ for the disorder standard deviation $\sigma=0.5$. The solid lines represent the exact results, while the symbols denote the results of the transfer-matrix calculations with an error of about 6%.

For small frequencies, the scattering amplitude is suppressed and the localization length is enhanced. The Ioffe-Regel criterion[26] for localization yields the physical picture of the problem. According to this picture, localization is stronger when $k\ell\sim1$, where $k$ is the wave number and $\ell$ is the mean-free path of elastic scattering (or of the carriers), which is of the order of the correlation length over which the mass fluctuates. If the correlation length (and, hence, $\ell$) of the fluctuations is decreased to zero, the criterion will be satisfied at small frequencies (long wavelengths). In other words, when there is a finite correlation length in the disorder, we have an upper cutoff in the short-wavelength limit (high frequencies) $\lambda_{min}<\xi$ so that for wavelengths shorter than $\lambda_{min}$, we have the so-called geometrical ray propagation.

If the correlation length approaches zero *continuously*, the corresponding short-wavelength limit will also approach smaller values and the localization length will also decrease in a larger domain of frequencies. In the limit of zero correlation length, which is the focus of the present work, there is no such limit and, therefore, no geometrical ray propagation

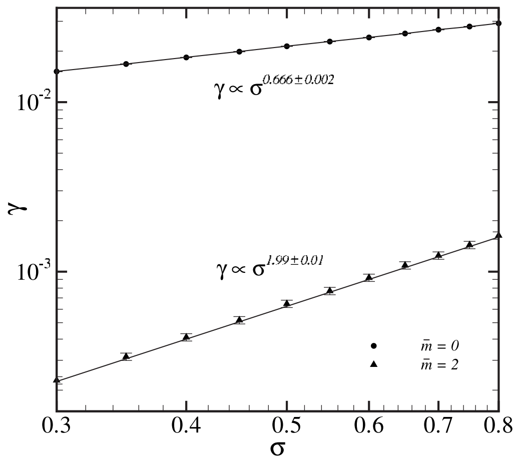


FIG. 2. Dependence of the Lyapunov exponent $\gamma$ on the standard deviations $\sigma$ of the disorder at frequency $\omega=0.2$. The solid lines represent the exact results, while the symbols show the results of the transfer-matrix calculations with an error of about 5% for the average density $m_a=1$ and 0.2% for $m_a=0$.

regime. The smallest characteristic frequency for an ordered chain is the cutoff frequency, which corresponds to the minimum wavelength, which is of the order of lattice spacing $a$. The cutoff frequency is approximately $\omega_c \simeq 2/a\sqrt{k_s/m_a}$, where $k_s$ is the spring constant. By low frequencies, we mean those that are smaller compared to $\omega_c$.

Figure 2 shows the dependence of the LE on $\sigma$ at frequency $\omega=0.2$, again computed both analytically and numerically. Once again, the numerical results are in complete agreement with the exact results given above.

In Fig. 3, we display the LE at larger frequencies, which indicates that, in the long-wavelength (small-frequency) limit, the small scale cutoff is irrelevant, and the continuous and discrete models yield the same results. For large frequencies, however, the spectrum of the continuous model [Eq. (2)] and that of the discrete model [Eq. (18)] differ significantly due to the presence of a high-frequency cutoff in the discrete model.

Figure 4 compares the exact DOS $\rho(\omega^2)$ of an ordered 1D lattice with the results of the numerical simulations, which

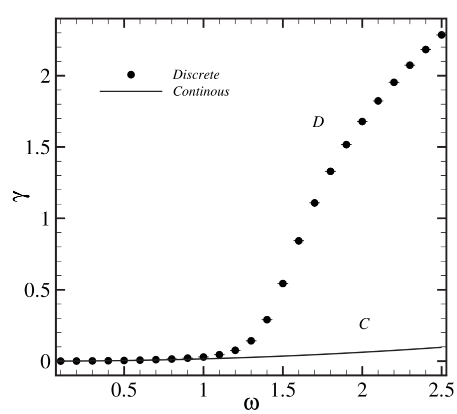


FIG. 3. Frequency dependence of the Lyapunov exponent $\gamma$ for (c) the continuous model and (d) the discrete model, computed by the transfer-matrix method for $m_a=2$ and $\sigma=0.5$.

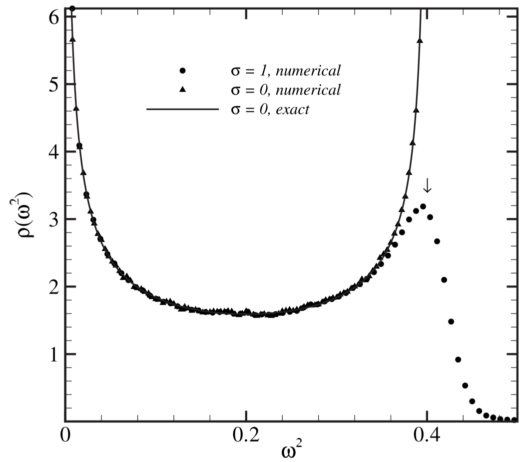


FIG. 4. Comparison of the density of states $\rho(\omega^2)$ of the discrete model with an average density $m_a=10$ and two values of the disorder standard deviations $\sigma$. The exact results are for the ordered 1D lattice, while the symbols denote the results of the numerical simulations of the model. The arrow indicates the position of the cutoff frequency.

were obtained for a single realization of a system with a size of $10^6$ by using the forced-oscillator method[27] (FOM), which is an efficient technique for computing the eigenvalues and eigenvectors of large matrices and, in particular, for calculating the DOS without direct diagonalization. The exact DOS of the ordered chain diverges at zero frequency and also at the cutoff frequency $\omega_c$. Then, by adding randomness to, for example, an average density $m_a=10$ and a standard deviation $\sigma=1.0$, the singularity at the upper band edge is rounded and a band tail appears, which leads to a maximum at the cutoff frequency $\omega_c$.

Figure 5 compares the computed DOSs for the continuous and discrete models. The numerical results for the discrete

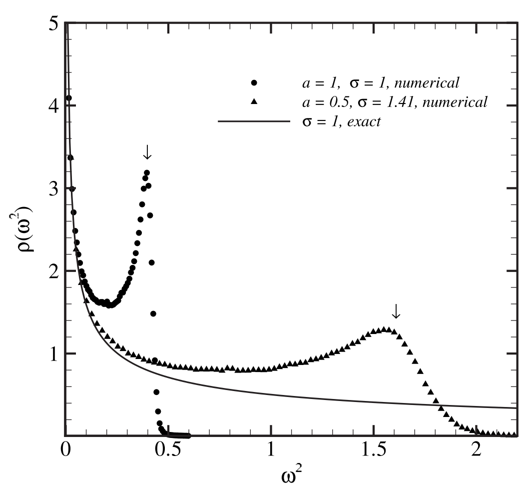


FIG. 5. Comparison of the density of states $\rho(\omega^2)$ of the continuous model (exact result, continuous curve) with the results of the numerical simulations for two values of the lattice constant $a$. The arrows indicate the cutoff frequencies. As $a$ decreases, the exact results for the continuous model and the numerical results become closer and the cutoff frequency shifts to higher values (shorter times).

model are for a single realization of a system with a size of $10^6$ using the FOM. Similar to the LE, the two DOSs agree at low frequencies (long times) but differ at intermediate and large frequencies (intermediate and short times). As Fig. 5 also indicates, the DOS of the discrete model has a maximum while that of the continuous model decreases monotonously. The differences between the two systems are expected: The agreement between the two DOSs should be in a limited range of the spectrum, not in the entire spectrum, because the DOS of the two models should be similar for frequencies that correspond to long times; that is, at times when the waves have sampled a large part of the lattice and the wavelengths are much larger than the lattice spacing $a$ of the discrete model. Therefore, near $\omega=0$ (long times), the two models have similar DOSs (and LE), but as the cutoff frequency $\omega_c$ is approached, they begin to differ and, in particular, the discrete model behaves completely differently from the continuous model.

If the lattice spacing $a$ is shrunk toward zero, then the cutoff frequency shifts to larger values (shorter times) and the range of frequencies over which the two models agree is enlarged. This is also shown in Fig. 5 for two values of the lattice spacing $a$. Note that changing the lattice constant $a$ in the discrete model entails changing the disorder strength $\sigma$. According to the correlation function [Eq. (3)], the discrete and continuous disorder strengths $\sigma$ are related through

$$\sigma_{\text{discrete}} = \frac{1}{\sqrt{a}}\sigma_{\text{continuous}}. \tag{22}$$

## IV. LYAPUNOV EXPONENT OF THE HARMONIC MAGNON MODES OF THE ONE-DIMENSIONAL HEISENBERG-MATTIS SPIN GLASSES

In this section, we consider an important feature of the dynamical properties of spin glasses and discuss its relation to the random wave equation considered above. Low energy excitations (spin waves), describing the dynamics of spin glasses, have an anomalous dispersion relation given by

$$\Omega \sim k^{3/2}, \tag{23}$$

where $\Omega$ is the magnon frequency and $k$ is the inverse characteristic length of the spin fluctuations.[28] The power law [Eq. (23)] is in contrast to the parabolic and linear dispersion relations of the ferromagnetic and antiferromagnetic systems,[28] respectively. Such an anomalous power law has also been reported[29] for the LE and the DOS of the model at low frequencies.

We consider the Heisenberg Hamiltonian for 1D spin chains with nearest-neighbor couplings as

$$H = \sum_{i,j} J_{ij} S_i \cdot S_j, \tag{24}$$

where $J_{ij}$ is distributed randomly according to a binary distribution function, $P(J_{ij}) = 1/2[\delta(J_{ij}-J)+\delta(J_{ij}+J)]$, and the coupling $J$ is a constant (set to unity in the following analysis). The Hamiltonian is then written in terms of the transverse operators, $S^{\pm}=S_x \pm iS_y$, as

$$H = \sum_{i,j} J_{ij}\left[\frac{1}{2}(S_i^+S_j^- + S_i^-S_j^+) + S_i^zS_j^z\right], \tag{25}$$

which leads to the following Heisenberg equation of motion for the transverse component:

$$i\hbar\frac{\partial S_i^+}{\partial t} = \sum_{j\neq i} J_{ij}\left\{\frac{1}{2}[S_i^+, S_i^+S_j^- + S_i^-S_j^+] + [S_i^+, S_i^zS_j^z]\right\}. \tag{26}$$

Using the relations $[S_i^+, S_j^-]=2\delta_{ij}S_i^z$ and $[S_i^z, S_j^{\pm}]=\pm\delta_{ij}S_i^{\pm}$ in Eq. (26), we obtain the following equation of motion:

$$i\hbar\frac{\partial S_i^+}{\partial t} = \sum_{j\neq i} J_{ij}(S_i^zS_j^+ - S_j^zS_i^+), \tag{27}$$

or, for 1D spin chains,

$$i\hbar\frac{\partial S_i^+}{\partial t} = -(J_{i,i+1}S_{i+1}^z + J_{i,i-1}S_{i-1}^z)S_i^+ + J_{i,i+1}S_i^zS_{i+1}^+ + J_{i,i-1}S_i^zS_{i-1}^+. \tag{28}$$

Now, assuming that the spin wave amplitudes are small (i.e., $S^x$, $S^y \ll S$), we may replace Eq. (28) with a linear equation, which is known as the Heisenberg-Mattis spin glass. This means that $S_i^z \approx \zeta_i S$, where $S$ is the length of the spin vectors and $\zeta_i \pm 1$. Using the identities $\zeta_i^2=1$ and $(S/\hbar J)\zeta_i\zeta_j J_{i,j}=-1$, we obtain

$$(2-\zeta_i\Omega)\mu_i = \mu_{i+1} + \mu_{i-1}, \tag{29}$$

where $\mu_i=\zeta_i S_i^+$. Here, we used $\mu_i(t)=\mu_i \exp(-i\Omega t)$.

Equation (29) is analogous to the discrete wave equation [Eq. (18)] with an average mass of zero if we set $\Omega=\omega^2$. In contrast to $\zeta_i$, however, $m_i$ in Eq. (18) can take any value. Due to the mapping between the two models, the exact results described in Sec. II are directly applicable to spin glasses. In particular, Eqs. (10) and (14), in the limit $m\to 0$, lead to

$$\gamma = \frac{6^{1/3}\sqrt{\pi}}{2\Gamma(1/6)}(\sigma\Omega)^{2/3}, \tag{30}$$

hence predicting the anomalous scaling of the LE with the magnon frequency $\Omega$, i.e., the one with an exponent of 2/3. Note that the numerical coefficient in Eq. (30) is not universal and depends on the form of the distribution function of $\zeta_i$. Here, we give its exact value for a continuous Gaussian distribution with variance $\sigma^2$. The positivity of the LE implies that all the magnon modes are localized in the 1D HM spin glasses. The exponent 2/3 in Eq. (30) is, indeed, the anomaly in the band edge of the Anderson model with a diagonal disorder, which has been studied extensively using the discrete model. For example, Derrida and Gardner[9] calculated the LE of the 1D Anderson model using a weak-disorder expansion and, therefore, their result is valid for a weak heterogeneity. Here, we reported the same anomalous exponent for a continuous model of wave propagation with a zero average mass.

## V. SUMMARY

The exact Lyapunov exponent (the inverse of the localization length) of the one-dimensional disordered wave equation in the low-frequency limit was derived a long time ago by Lifshits *et al.*[17] We derived the corresponding expression for the high-frequency limit. The LE and the DOS of the random wave equation in a 1D lattice were computed by numerical simulations. For small frequencies, the simulation results are in complete agreement with the exact results but deviate from them at high frequencies. The difference is due to the existence of a cutoff frequency in the discrete system. Since a discrete system can be thought of as one in which the correlation length is the same as the lattice constant, the difference also distinguishes such lattices from continuous random media in which the correlation length is zero and for which the exact results have been derived.

By deriving a mapping between the equation of motion for the dynamics of the tranverse modes at zero temperature of the Heisenberg-Mattis model of 1D spin glasses and the discretized wave equation with nonzero density, we also obtained an exact expression for the LE of the magnon modes of the spin glasses, which indicates anomalous scaling of the LE with the frequency.

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