Revised Version IPM-95-110, hep-th/9507166

The Alf'ven Effect and Conformal Field Theory

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Abstract

Noting that two-dimensional magnetohydrodynamics can be modeled by conformal field theory, we argue that when the Alf'ven effect is also taken into account one is naturally lead to consider conformal field theories, which have logarithmic terms in their correlation functions. We discuss the implications of such logarithmic terms in the context of magnetohydrodynamics, and derive a relationship between conformal dimensions of the velocity stream function, the magnetic flux function and the Reynolds number.

1 - Introduction

There has been some work on modelling turbulence in two dimensional fluids by conformal field theory (CFT) [1-7]. Ferretti et al. [6] have generalized Polyakov's method [1] to the case of two dimensional magnetohydrodynamics (2D - MHD). We have argued that the existence of a critical dynamical index is equivalent to the Alf'ven effect [7] i.e. the equipartition of energy between velocity and magnetic modes [8]. The Alf'ven effect, reduces the number of candidate conformal field theories, but also it implies that the velocity stream function ϕ and the magnetic flux function ψ should have similar scaling dimensions. To reduce the number of candidate conformal field theories other condition on 2D-MHD, have been imposed by Coceal and Thomas [9]. Gurarie [10] has argued that although in unitary minimal models two primary fields with the same dimension do not occur , such a situation can occur in non minimal CFTs. In such conformal field theories, it has been shown [10] that the correlator of two fields, has a logarithmic singularity.

$$< \psi(r)\psi(r') > \sim |r - r'|^{-2h_\psi} \log|r - r'| + \dots \tag{1}$$

Examples of such theories have been studied by Gurarie [10] , Saleur [11], Rozansky and Saleur in connection with the Wess - Zumino - Witten model on the super group GL(1,1) [12] and Bilal and Kogan in connection with the gravitational dressed CFT [13,14]. This paper is organised as follows; in section two we give a very brief summary of magnetohydrodynamics and the Alf'ven effect. In section 3 we discuss the implication of the logarithmic divergence and candidate CFT models are given in section 4 .

2 - The Alf'ven effect and conformal field theory.

The incompressible two dimensional magnetohydrodynamic (2D - MHD) system has two independent dynamical variables, the velocity stream function ϕ and the magnetic flux function ψ. These obey the pair of equations [15],

$$\frac{\partial \omega}{\partial t} = -e_{\alpha\beta}\partial_\alpha \phi \partial_\beta \omega + e_{\alpha\beta}\partial_\alpha \psi \partial_\beta J + \mu \bigtriangledown^2 \omega \tag{2}$$

$$\frac{\partial \psi}{\partial t} = -e_{\alpha\beta}\partial_\alpha \phi \partial_\beta \psi + \eta J \tag{3}$$

where the vorticity $\omega = \bigtriangledown^2 \phi$ and the current $J = \bigtriangledown^2 \psi$. The two quantitiesy μ and η are the viscosity and molecular resistivity, respectively. The velocity and magnetic fields are given in terms of ϕ and ψ :

$$V_\alpha = e_{\alpha\beta}\partial_\beta \phi \tag{4}$$

$$B_\alpha = e_{\alpha\beta}\partial_\beta \psi \tag{5}$$

and $e_{\alpha\beta}$ is the totally antisymmetric tensor, with $e_{12} = 1$. Chandrasekhar [7] has shown that the Alf'ven effect or the equipartition of energy between velocity and magnetic modes requires $V_k^2 = \alpha B_k^2$, with α of order unity. In fact he finds $\alpha = 1.62647$ for 2D - MHD. We [8] have argued that the existence of a critical dynamical index for 2D - MHD, implies the Alf'ven effect and if the conformal model holds, this implies the equality of scaling dimensions of ϕ and ψ :

$$h_\phi = h_\psi \tag{6}$$

Here the criteria of Gurarie [10] are satisfied and these two fields are logarithmically correlated. According to Gurarie [10], the operator product expansion of two fields A and B, which have two fields ϕ and ψ of equal dimension in their fusion rule [16] , has a logarithmic term:

$$A(z)B(0) = z^{h_\phi - h_A - h_B}\{\psi(0) + \ldots + \log z(\phi(0) + \ldots)\} \tag{7}$$

to see this it is sufficient to look at four point function :

$$< A(z_1)B(z_2)A(z_3)B(z_4) >\sim \frac{1}{(z_1-z_3)^{h_A}} \frac{1}{(z_2-z_4)^{h_B}} \frac{1}{[x(1-x)]^{h_A+h_B-h_\phi}} F(x) \qquad (8)$$

Where the cross ratio x is given by :

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} \qquad (9)$$

In degenerate models $F(x)$ satisfies a second order linear differential equation. Therefore a solution for $F(x)$ can be found in terms of a series expansion :

$$F(x) = x^\alpha \sum a_n x^n \qquad (10)$$

In the next section we will show that the existence of two fields with equal dimension in OPE of A and B is equivalent to the secular equation for α having coincident roots, in which case two independent solutions can be constructed according to :

$$\sum b_n x^n + \log x \sum a_n x^n \qquad (11)$$

Now consistency of equation (12) and (8) requires :

$$< A(z_1)B(z_2)\psi(z_3) >=< A(z_1)B(z_2)\phi(z_3) > \{\log \frac{(z_1-z_2)}{(z_1-z_3)(z_2-z_3)} + \lambda\} \qquad (12)$$

$$< \psi(z)\psi(0) >\sim \frac{1}{z^{2h_\psi}}[\log z + \lambda^{'}] \qquad (13)$$

$$< \psi(z)\phi(0) >\sim \frac{1}{z^{2h_\phi}} \qquad (14)$$

where λ and $\lambda^{'}$ are constants. Eq.(14) is consistent with the findings of Gurarie [17] and Polyakov [18] which shows that the probability distribution of such correlation functions is different from the Gibbs distribution since for the Gibbs distribution we should have $< \psi(z)\phi(0) >= 0$.

Let us now consider the action of $SL(2,C)$, on the correlator $< \psi(z_1)\psi(z_2) >$. The generator of $SL(2,C)$, $(L_0, L_{\pm 1})$, act on this correlator as follows;

$$\begin{aligned}
&L_{-1} < \psi(z_1)\psi(z_2) >= 0\\
&L_0 < \psi(z_1)\psi(z_2) >= -2|z_1 - z_2|^{-2h_\phi}\\
&L_{+1} < \psi(z_1)\psi(z_2) >= -2|z_1 - z_2|^{-2h_\phi}|z_1 + z_2|
\end{aligned} \tag{15}$$

By simple algebra one observes that $< \psi(z_1)\psi(z_2) >$ is invariant under$(L_{-1}, L_0^2, L_+ L_0)$ and also $L_0 < \psi(z_1)\psi(z_2) >$ is itself invariant under the action of $SL(2,C)$.
Thus $L_0 < \psi(z_1)\psi(z_2) >$ behaves like an ordinary CFT correlation function. Thus we may solve the resulting first order differential equation for $< \psi(z_1)\psi(z_2) >$, which naturally leads to a logarithmic singularity. This result is compatible with the finding in [10] that this type of operator together with ordinary primary operators form the basis of the Jordan cell for the operator L_0. This fact allows us to find higher-order correlation functions for the operator ψ.

3- The Infrared problem and The Energy Spectrum:

The presence of logarithmic terms requies a reconsideration of the infrared problem. The k-representation of the correlation is;

$$< \psi(k)\psi(-k) >= |k|^{-2-2|h_\phi|}[C_1 + \log k] \tag{16}$$

which is divergent in the limit of $k \to 0$. One can set some cut-off in the k-space to remove this divergence :

$$\begin{aligned}
< \psi(x)\psi(0) > &= \int_{k>\frac{1}{R}}^{\infty} k^{-2-2|h_\phi|}[C + \log k]e^{ik\cdot x}d^2k\\
&\sim R^{2|h_\phi|}(\log R + C') - x^{2|h_\phi|}(C' + \log X) + \ldots)
\end{aligned} \tag{17}$$

where R is the large scale of the system. It seems that it is natural to add some condensate term [1] in momentum space to cancel the infrared divergence. The energy spectrum for this type of correlation, is

$$E(k) \simeq k^{-2|h_\phi|+1}(C + \log k) \tag{18}$$

which has a logarithmic singularity at the limit of $k \to 0$. This spectrum is compatible with the results of Ref. [19] where it has been shown that, one loop correction to the energy spectrum gives a logarithmic contribution to the energy spectrum.

4- Finding a Candidate Conformal Field Theory.

The question is which types of conformal field theory may be used for modelling 2D-MHD turbulence, provided we take into account the Alf'ven effect as well as the cascade of the mean square magnetic potential. At first glance, we cannot use the minimal models. For a fixed (p, q) all the primary fields in minimal models have different dimensions, thus eq.(6) is never satisfied. To see this let us look at the conformal dimension $h_{m,n}$ of a given primary field $\phi_{m,n}$

$$h_{m,n} = \frac{1}{4pq}[(mp - nq)^2 - (p - q)^2] \tag{19}$$

with $1 \leq q \leq p$,$1 \leq m \leq q - 1$ and $1 \leq n \leq p - 1$. Simple algebra shows that if two fields $\phi_{m,n}$ and $\phi_{m',n'}$ have the same dimensions $h_{m,n} = h_{m',n'}$, then we must have

$$\frac{m \pm m'}{n \pm n'} = \frac{p}{q} \tag{20}$$

And since p and q are coprime, eq.(20) is never satisfied. However all is not lost, one can find non unitary minimal models where two primary fields have almost equal conformal dimensions. The table of CFT models which are nearly consistent with (6) is given in reference

[8]. Two primary fields with almost equal conformal dimensions can mimic logarithmic correlators for a restirected range.

According to ref.[20], the hypergeometric equation govrning the correlator of two fields in whose OPE two other fields ψ and ϕ with conformal dimesions h_ψ and $h_\psi + \epsilon$ appear, admits two solutions;

$$ {}_2F_1(a, b, c, x) \tag{21} $$

$$ x^\epsilon \ {}_2F_1(a + \epsilon, b + \epsilon, c + 2\epsilon, x) \tag{22} $$

where a, b and c are sums of conformal dimension. Clearly in the limit of $\epsilon \to 0$ these two solutions coincide. Another independent solution exists, it involves logarithms and can be generated by standard methods [21]. Therefore expanding the above solutions neae $\epsilon = 0$ logarithmic behaviour is obtained.

Construct two fields Φ_+ and Φ_-,

$$ \Phi_\pm = \psi \pm i\lambda\phi \tag{23} $$

where λ is a constant with dimension ϵ. Then the correlators are

$$ < \Phi_+(z)\Phi_+(0) > \simeq z^{-2h_\psi} \ln z \tag{24} $$

and

$$ < \Phi_+(z)\Phi_-(0) > \simeq z^{-2h_\psi} \tag{25} $$

provided z lies in the range:

$$ a \ll z \ll R \tag{26} $$

where a is the dissipation range, this gives:

$$\lambda \simeq R^{\epsilon} \qquad\qquad \epsilon \leq \frac{1}{5/2 \ln R_e} \tag{27}$$

where R_e is the typical Reynold's number of system and we have used the relation $a \simeq R{R_e}^{-5/2}$, which can easily be seen using dimensional arguments. For example turbulence of up to $R_e \sim 10^{12}$ may be describ by the minimal model $(6, 35)$, which has $\epsilon = 1/70$.

The above is of course an approximate argument and the approximation improves as ϵ tends towards zero or the central charge tends towards unity. If we insist on exact logarithmic correlators we need to consider other CFT 's , probably with the effective central charge equal to unity [22] and also see [23]. Work in this direction is under progress.

Acknowledgements The authors are grateful to S. Parvizi for stimulating discussions, and wish to thank Werner Nahm and the referee for pointing out an error in the earlier version of this article.

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