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Zero Tension Kardar-Parisi-Zhang Equation in $(d+1)$–Dimensions

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The joint probability distribution function (PDF) of the height and its gradients is derived for a zero tension $d+1$-dimensional Kardar-Parisi-Zhang (KPZ) equation. It is proved that the height‘s PDF of zero tension KPZ equation shows lack of positivity after a finite time t_c. The properties of zero tension KPZ equation and its differences with the case that it possess an infinitesimal surface tension is discussed. Also potential relation between the time scale t_c and the singularity time scale $t_{c,\nu\to 0}$ of the KPZ equation with an infinitesimal surface tension is investigated.

KEY WORDS: Probability distribution function; the Kardar-Parisi-Zhang equation; strong coupling limit; singularity time scale.

1. INTRODUCTION

Studying the morphology, formation and growth of interfaces has been one of the recent interesting fields of study because of its high technical and rich theoretical advantages. On account of the disorder nature embedded in the surface growth, stochastic differential equations have been used as a suitable tool for understanding the behavior of various growth processes. Such equations typically describe the interfaces at large length scales, which means that the short length scale details has been neglected in order to derive a continuum equation by focusing on the

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coarse grained properties. A great deal of recent theoretical modeling has been started with the work of Edward and Wilkinson[1] describing the dynamics of height fluctuations by a simple linear stochastic equation. By adding a new term proportional to the square of the height gradient, Kardar, Parisi and Zhang made an appropriate description for lateral interface growth[2]. The $d+1$-dimensional forced KPZ equation is written as

$$\frac{\partial h}{\partial t} - \frac{\alpha}{2}(\nabla h)^2 = \nu \nabla^2 h + f \tag{1}$$

where $h(\mathbf{x}, \mathbf{t})$ specifics the surface height at point $\mathbf{x}$ (d-dimensional vector) and $\alpha \geqslant 0$. The force f is a zero mean, statistically homogeneous, white in time, Gaussian process which it's covariance would be

$$\langle f(\mathbf{x}, t) f(\mathbf{x}', t') \rangle = 2D_0 D(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{2}$$

Typically the spatial correlation of the forcing is considered to be a delta function, mimicking the short length correlation. Here the spatial correlation is considered as

$$D(\mathbf{x}-\mathbf{x}') = \frac{1}{(\pi\sigma^2)^{d/2}} \exp\left(-\frac{(\mathbf{x}-\mathbf{x}')^2}{\sigma^2}\right) \tag{3}$$

where σ is much less than the system size L, i.e. $\sigma \ll L$, which represents a short range character for the random forcing. It is useful to rescale the KPZ equation as $h' = h/h_0$, $\mathbf{r}' = \mathbf{r}/\mathbf{r_0}$ and $t' = t/t_0$. If we let $h_0 = (\frac{D_0}{\nu})^{1/2}$ and $t_0^2 = \frac{r_0}{\nu}$, where r_0 is a characteristic length, all of the parameters can be eliminated, except the coupling constant $g = \frac{\alpha^2 D_0}{\nu^3}$. The limit $g \to \infty$ (or zero tension limit, $\nu \to 0$), is known as the strong coupling limit.[13,14] Although originally, this equation appeared as a model for surface growth,[3] it is mostly used today in polymer physics,[4] Burgers turbulence,[5] nonlinear acoustics[6] and cosmology,[7] etc.

The nonlinearity of the KPZ equation includes the possibility of singularity formations in a finite time as a result of the local minima instability. Meaning that there is a competition between the diffusion smoothing effect (the Laplacian term), and the enhancement of non-zero slopes. Let us mention the main properties of the KPZ equation for the cases $\nu \to 0$ and $\nu = 0$. In the *zero tension limit* ($\nu \to 0$) the KPZ equation has the following properties: (i) the unforced KPZ equation develops singularities for given dimensions. In one spatial dimension the sharp valleys are developed in a finite time $t_{c,\nu\to 0}$. The geometrical picture consists of a collection of sharp valleys intervening a series of hills in the stationary state[8]. In two spatial dimensions the KPZ equation develops three types of singularities in finite time. The first singularities are sharp valley lines with finite lengths, which the height gradients are discontinues while crossing the valley lines. The second type are the end points of the sharp valley lines. As time goes on these sharp valley lines hit each other and the crossing point of two valley lines produces a valley node. Generically these end points disappear at large time scales and only a network of sharp valley lines will survive.[5,8,9] In three and higher dimensions, the structure of the singularities can be more complicated. For instant, in three dimensions the singularities are, in the language of Burgers equation, shock surfaces, its boundaries, the intersection line of two shock surfaces and finally the point which three shock surfaces meet each other. The height gradients are discontinues while crossing the shock surfaces. A complete classification of the KPZ singularities, by considering the metamorphosis of singularities as time elapses, has been done in ref. 10. (ii) Similarly, for white in time and smooth in space forcing, in the zero tension limit singularities will be developed in a finite time in any given spatial dimensions. For instance, in two spatial dimensions the sharp valley lines are smooth curves where in the stationary state the sharp valley lines produces a curvilinear hexagonal lattice[9,15] (see Fig. 1). In three dimensions it can be shown that the shape of the polyhedra tiling the space can not be determined uniquely, nevertheless, the minimal polyhedra in $d=3$ will have 24 vertices.[9]

The KPZ equation with vanishing surface tension $(\nu=0)$ produces multi-valued solution after time scale $t_{c,\nu=0}$.[5] In Fig. 2 we demonstrate the multi-valued solution of the unforced zero tension KPZ equation in two dimensions. We have used the Lagrangian method to simulate the KPZ equation with $\nu=0$ and initial condition $h(x,y,0)=sin(x)sin(y)$[5]. The sinusoidal function is a typical initial condition and has been used only for simplicity. With this initial condition the time scale that the KPZ equation produces the multivalued solution can be found exactly as $t_{c,\nu=0}=1$.[5] As shown in Fig. 2-a, it is evident that for time scales $t<t_{c,\nu=0}$ the height field is single-valued. At the time scale $t=t_{c,\nu=0}$ the height field become singular (see Fig. 2-b). Finally in Fig. 2-c we have plotted the height field for time scale $t>t_{c,\nu=0}$ in which the height is multi-valued. The singularities in the limit $\nu\to 0$ can be constructed from multi-valued solutions of the KPZ equation with $\nu=0$ by Maxwell cutting rule,[5] which makes the discontinuity in the derivative of height field. Indeed the Maxwell cutting rule states, that for $\mathbf{x}$'s which the height field $h(x,t)$ becomes multi-valued, the physical solution can be chosen so that, $h_{ph}(\mathbf{x},t)=Max\{h_1(\mathbf{x},t),h_2(\mathbf{x},t),\dots,h_N(\mathbf{x},t)\}$, where N is the number of the multiple solutions of the field $h(\mathbf{x},t)$ at position $\mathbf{x}$.

In this paper, an exact master equation is derived from the *zero tension* $(\nu=0)$ KPZ equation for the joint probability distribution function

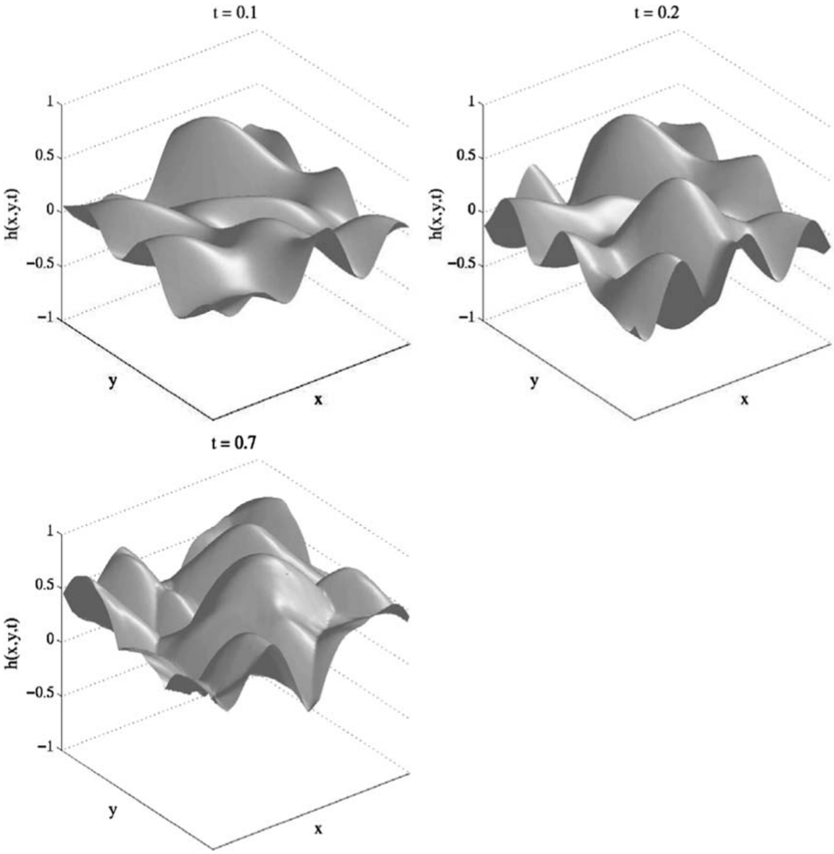

Fig. 1. Different time snapshots of height fields of randomly driven two dimensional KPZ equation in the zero tension limit ($\nu \to 0$). Two upper snapshots are belong to the height fields before singularity time scales $t_{c,\nu\to 0}$. In the lower figure which is for time scales $t > t_{c,\nu\to 0}$, the field $h(x, y)$ is not differentiable in singularity curves.(12) In this simulation the relation between the forcing length scale σ and the sample size L, is $\sigma \simeq L/3$. The forcing strength D_0 is equal to unity.

(PDF) of height and its gradients. The master equation enables us to determine the time evolution of the PDF of $h - \bar{h}$ and all of the moments $\langle (h - \bar{h})^n \rangle$. It is proved that the derived height‘s PDF for the $\nu = 0$ case, shows lack of positivity after a finite time t_c. Potential relation between the t_c and the singularity time scale ($t_{c,\nu\to 0}$) of the KPZ equation having an infinitesimal surface tension is discussed. Details of calculations are presented in the appendices A, B and C.

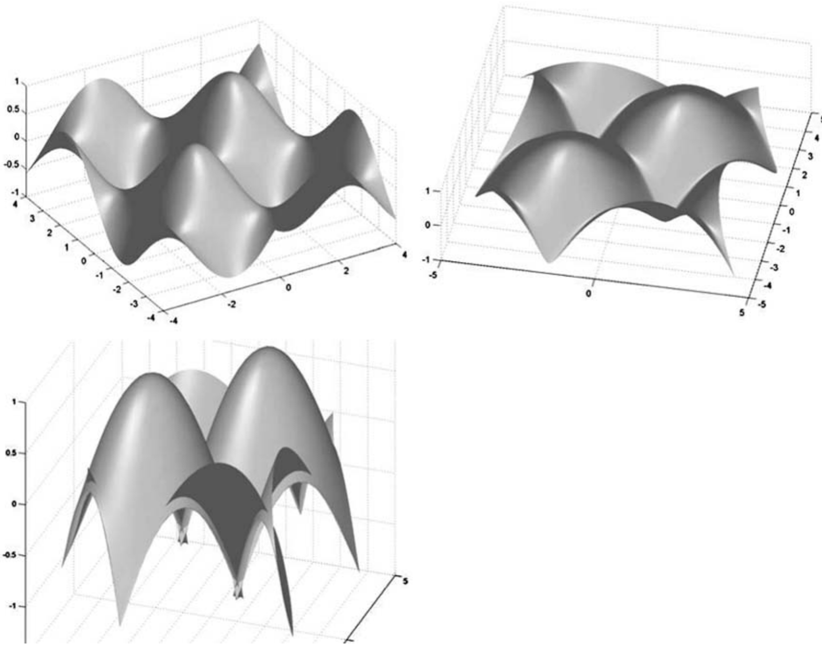

Fig. 2. Different time snapshots of height fields of the zero tension ($\nu = 0$), unforced two dimensional KPZ equation with initial condition $h(x, y, 0) = sin(x)sin(y)$. In this case the time scale $t_{c,\nu=0}$ is equal 1. In the upper figure the height field is single valued (for time scale $t < t_{c,\nu=0}$). In the middle we demonstrate the height field in time scale $t = t_{c,\nu=0}$ which is the singularity time scale. In the lower figure, which is for time scales $t > t_{c,\nu=0}$, the field $h(x, y)$ is multi-valued.

2. JOINT PDF OF HEIGHT AND ITS GRADIENTS FOR ZERO TENSION KPZ EQUATION

Let us define $P(\tilde{h}, u_i, p_{ij}, t)$ as the joint PDF of $\tilde{h} = h - \bar{h}$, $u_i = h_{x_i}$ and $p_{ij} = h_{x_i x_j}$, where $i, j = 1, 2, \ldots, d$. Using the zero tension KPZ equation, it is shown in Appendix A, that the $P(\tilde{h}, u_i, p_{ij}, t)$ satisfies the following equation

$$\begin{aligned} P_t &= \gamma(t) P_{\tilde{h}} + \frac{\alpha}{2} \sum_l u_l^2 P_{\tilde{h}} - \alpha(d+2) \sum_l p_{ll} P \\ &\quad -\alpha \sum_{l,k \leqslant m} p_{lk} p_{lm} P_{p_{km}} + k(0) P_{\tilde{h}\tilde{h}} - k''(0) \sum_l P_{u_l u_l} \\ &\quad +2k''(0) \sum_l P_{\tilde{h} p_{ll}} + k''''(0) \sum_{l \leqslant k} P_{p_{lk} p_{lk}} \\ &\quad -2k''''(0) \sum_{l<k} P_{p_{ll} p_{kk}}. \end{aligned} \tag{4}$$

where $\gamma(t)=\bar{h}_t$, $k(\mathbf{x}-\mathbf{x}')=2D_0D(\mathbf{x}-\mathbf{x}')$, $k''(0)=k_{x_ix_i}(0)$ and $k''''(0)=k_{x_ix_ix_ix_i}(0)$. Eq. (4) enables us to calculate the joint PDF $P(\tilde{h},u_i,p_{ij},t)$ and all the moments $S_n=\langle\tilde{h}^n\rangle$ in d-dimensions. Various moments has been calculated explicitly for the three-dimensional case in Appendix B. To derive a closed expression for $P(h-\bar{h},u_i,t)$ one needs to know the moments such $\langle h^n u_i^m u_j^l p_{ij}\rangle$. As appeared in Appendix B, using Eq. (4) one can show that such moments are identically zero.[11] Using the identity mentioned above, it can be shown that $P(\tilde{h},u_i,t)$ has the following expression (see Appendix C);

$$P(\tilde{h},u_i,t)=\int\frac{d\lambda}{2\pi}\prod_i^d\frac{d\mu_i}{2\pi}\exp(i\lambda(h-\bar{h}(t))+i\sum_l^d\mu_lu_l)\times Z(\lambda,\mu_i,t) \tag{5}$$

where,

$$Z(\lambda,\mu_i,t)=F_1(\lambda,\mu_1)F_2(\lambda,\mu_2)...F_d(\lambda,\mu_d)\exp\left(-\lambda^2k(0)t\right) \tag{6}$$

and,

$$\begin{aligned}F_j(\lambda,\mu_j,t)&=(1-\tanh^2(\sqrt{2ik_{xx}(0)\alpha\lambda}t))^{-\frac{1}{4}}\\ &\exp\left[-\frac{i}{2}\alpha k''(0)\lambda t^2-\frac{1}{2}i\mu_j^2\sqrt{\frac{2ik_{xx}(0)}{\alpha\lambda}}\tanh(\sqrt{2ik_{xx}(0)\alpha\lambda}t)\right].\end{aligned} \tag{7}$$

Indeed $Z(\lambda,\mu_i,t)$ is the generating function of the $\tilde{h}$ and u_i's. Therefore expanding Eq. (6) in powers of λ, all the moments S_n can be derived. For instance, the first five moments are as follows

$$\begin{aligned}
\langle\tilde{h}^2\rangle&=\left(\frac{k^2(0)}{\alpha k''(0)}\right)^{\frac{2}{3}}\left[-\left(\frac{d}{3}\right)\left(\frac{t}{t^*}\right)^4+2\frac{t}{t^*}\right]\\
\langle\tilde{h}^3\rangle&=-\frac{8d}{15}\left(\frac{k^2(0)}{\alpha k''(0)}\right)\left(\frac{t}{t^*}\right)^6\\
\langle\tilde{h}^4\rangle&=\left(\frac{k^2(0)}{\alpha k''(0)}\right)^{\frac{4}{3}}\left[\left(\frac{1}{3}d^2-\frac{136}{105}d\right)\left(\frac{t}{t^*}\right)^8-4d\left(\frac{t}{t^*}\right)^5+12\left(\frac{t}{t^*}\right)^2\right]\\
\langle\tilde{h}^5\rangle&=-\left(\frac{k^2(0)}{\alpha k''(0)}\right)^{\frac{5}{3}}\left[\frac{16}{9}d\left(\frac{248}{105}-d\right)\left(\frac{t}{t^*}\right)^{10}+\frac{32}{3}d\left(\frac{t}{t^*}\right)^7\right]
\end{aligned}$$

where $t_*=\left(\frac{k(0,0)}{\alpha^2k''^2(0,0)}\right)^{1/3}$.

The important content of the exact expressions derived above is that through them the time scale that the height's PDF $P(h-\bar{h},t)$ lacks positivity condition after t_c. The positivity of PDF means that all the even moments of $\langle (h-\bar{h})^{2n}\rangle$ must be positive. In fact the above moment relations indicate that different even order moments become *negative* in some distinct characteristic time scales. Closer looking at the even moment relations reveals that the higher the moments are, the smaller their characteristic time scales become such that asymptotically tends to $t_c = a_d t_*$ for very large even moments. The coefficients a_d are of order of unity.[8,11] Indeed it can be shown that after time scale t_c the *right tail* of the probability distribution function (PDF) of height fluctuations (i.e. $P(h-\bar{h},t)$) is going to become negative, which is reminiscent of the singularity creation. In what follows we argue that the two time scales t_c and $t_{c,\nu=0}$ are related each other and $t_c \simeq t_{c,\nu=0}$.[8,11]

We note that the Eq. (6) has the property that $Z(0,0,0,t) = 1$ which means that $\int_{-\infty}^{+\infty} P(h-\bar{h},u,v;t)d(h-\bar{h})dudv = 1$ for *every time* t $(0 \preceq t \prec \infty)$. So the PDF of $h-\bar{h}$ and its derivatives is always normalizable to unity. In the limit of $\nu = 0$ after $t_{c,\nu=0}$ the height field becomes multi-value on the valleys, which is related to the left tail of the $P(h-\bar{h})$. The multiplicity of height field on valleys, on which the height difference $h-\bar{h}$ is mostly negative, increases the probability measure in left tail of the PDF. Therefore to compensate the exceeded measure related to the multi-valued solutions the right tail of the PDF tails should become negative. Therefore one concludes that $t_c \simeq t_{c,\nu=0}$. On the other hand as mentioned in the introduction the singularities in the limit $\nu \to 0$ can be constructed from multi-valued solutions of the KPZ equation with $\nu = 0$ by Maxwell cutting rule,[5] which makes the discontinuity in the derivative of height field. Therefore the time scale that the zero tension KPZ equation produces multi-valued solutions is the same as the time scale of singularity formation in KPZ equation with infinitesimal surface tension. So $t_{c,\nu=0} = t_{c,\nu\to 0} \simeq t_c$, where $t_c = a_d t_*$ and $t_* = \left(\frac{k(0,0)}{\alpha^2 k''^2(0,0)}\right)^{1/3}$. The t_c scales with σ as $\sigma^{\frac{d+4}{3}}$. Hence the smaller the σ, the shorter the time scale of singularity creation.

Taking into account that $\alpha > 0$ and $k''(0,0) < 0$, the odd order moments S_{2m+1} are positive in time scales before formation of multi-valued solution. It means that the probability density function $P(h-\bar{h},t)$ in this time regime is positively skewed. Therefore the probability distribution functions of height difference has a non zero skewness as it evolves in time, at least up to the time scale where the multivalued solutions are formed. In Fig. 3-a, using the Eqs. (4)–(7), we have numerically sketched the PDF evolution in time for zero tension $3+1$-dimensional

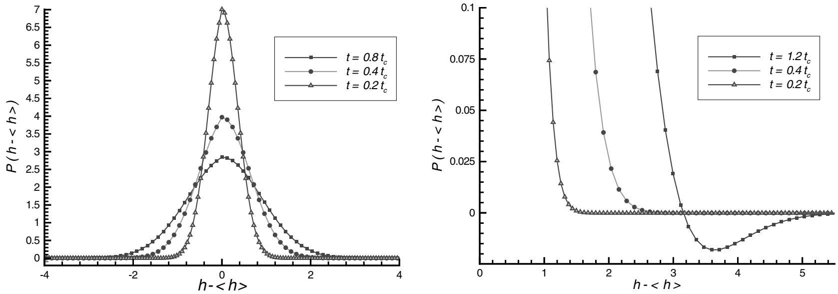

Fig. 3. In the upper graph the time evolution of PDF of $h-\bar{h}$ before formation of multivalued solutions at $0.2t_c$, $0.4t_c$ and $0.8t_c$ is numerically obtained. Lower graph shows the right tails of the PDF of $h-\bar{h}$ for $0.2t_c$, $0.4t_c$ and $1.2t_c$ corresponding to time scales before and after formation of multivalued solutions.

KPZ equation. As the system evolves in time, the formation of the multivalued solutions leads to the negativity of the right tail in the PDF. Also the evolution of PDF right tail for time scales before and after t_c can be checked in Fig. 3-b.

Now we would like to add a few comments on the equation governing the PDF of $h-\bar{h}$. As shown in Eq. (4), for zero tension KPZ equation, the $P(h, h_x, h_{xx}; t)$ satisfies a closed equation. Adding an infinitesimal surface tension ($\nu \neq 0$) to the KPZ equation, the PDF equation will no longer be closed. Here we have proved that the height's PDF of zero tension KPZ equation lacks the positivity condition after the finite time scale t_c. The situation is similar to the results given in ref. 16, where the derived PDF of velocity increments of *invisid* Burgers equation is not positive definite. In 1+1 dimensions, it has been already proved that adding an infinitesimal surface tension will guarantee the positivity of the PDF of $h-\bar{h}$.[8,17]

The same argument to find the singularity time scale i.e. t_{c0}, can be applied to the problem of decaying tensionless KPZ equation with random initial condition. We use the following probability density functional for initial height field ($h_0(x)$ and its spatial derivative ($u_0(x)$ in 1 + 1 dimension:

$$P[h_0(x), u_0(x)] \propto \exp\left(-\int dx dx' h_0(x) B(x-x') h_0(x') + \int dx dx' u_0(x) B''(x-x') u_0(x')\right), \qquad (8)$$

where the equality will be hold by a normalization constant and $B''(x) = B_{xx}(x)$. The initial distribution (8) shows that initial height field and its derivative are both Zero mean, statistically homogeneous, Gaussian processes and spatially correlated with covariance:

$$\langle h_0(x)h_0(x')\rangle = 2B(x-x') \tag{9}$$

$$\langle u_0(x)u_0(x')\rangle = -2B''(x-x') \tag{10}$$

$$\langle h_0^n(x)u_0^m(x')\rangle = \langle h_0^n(x) >< u_0^m(x')\rangle. \tag{11}$$

We set $B(x)$ to be Gaussian function with standard deviation σ_0:

$$B(x-x') = \frac{1}{\sqrt{(\pi\sigma_0^2)}} \exp\left(-\frac{(x-x')^2}{\sigma_0^2}\right), \tag{12}$$

σ_0 is the correlation length of initial height field. Now using the KPZ equation and its derivative we can derive the equations governing the time evolution of height and height derivative moments. This procedure has been represented in Appendix B (by setting $f=0$). Since we are interested in functional dependence of the time at which the singularities begin to form, it is sufficient to derive the second moment of height which can be easily found as:

$$\langle \tilde{h}^2\rangle = -\frac{1}{2}\alpha^2\langle u_0^2\rangle^2 t^2 + \langle \tilde{h_0}^2\rangle. \tag{13}$$

Eq. (13) shows that the second height moment would be zero at time $t_0^* = \left(\frac{\langle 2\tilde{h_0}^2\rangle}{\alpha^2\langle \tilde{u_0}^2\rangle^2}\right)^{1/2}$. Evaluating $\langle \tilde{h_0}^2\rangle$ and $\langle \tilde{u_0}^2\rangle$ from Eq. (8) and using Eq. (12) we see that t_0^* scales as $\sigma_0^{5/2}$. Due to the fact that the time scale of singularity formation (t_{c0}) is proportional to t_0^* by a constant of order of unity, the same conclusion is true for t_{c0}. Generalization of the above argument to higher dimensions is straightforward. It can be easily shown that $t_{c0} \sim \sigma_0^{\frac{d+4}{2}}$. This means that the smaller their characteristic initial length scales cause the singularities to be produced at smaller time scales.

To summarize, we obtain some results on the problem of KPZ equation in $d+1$ dimensions with a Gaussian forcing, which is white in time and short-range-correlated in space. In the non-stationary regime, where the singularities are not yet developed, we find an exact form for the generating function of the joint fluctuations of height and height gradients. Starting from a flat initial condition, we determine the time scale of the singularity formation and the exact functional form of the time dependence in the height difference moments at any given order. We note that if there is no singularity in the height field, the time evolution of the height's PDF of tensionless KPZ and the KPZ equation with infinitesimal surface tension are identical. This is the reason that the term $limit_{\nu\to 0}(\nu\nabla^2 h)$ is equal to zero for time scales before the singularity time scale t_c. We were able to give the solution of the PDF's equation for time scales before t_c.

We believe that the analysis followed in this paper is also suitable for the zero temperature limit in the problem of directed polymer in a random potential with short range correlations.[4]

APPENDIX A

In this appendix we prove the Eq. (4). Define generating function $Z(\lambda, \mu_i, \eta_{ij}, x_i, t) = \langle \Theta(\lambda, \mu_i, \eta_{ij}, x_i, t)\rangle$ for the fields $\tilde{h} = h - \bar{h}$, $u_i = h_{x_i}$ and $p_{ij} = h_{x_i x_j}$. The λ, μ_i and η_{ij} are the sources of $\tilde{h} = h - \bar{h}$, $u_i = h_{x_i}$ and $p_{ij} = h_{x_i x_j}$, respectively and $i, j = 1, 2, \ldots, d$.

The explicit expression of Θ is as follows:

$$\Theta = \exp(-i\lambda(h(x,y,z,t) - \bar{h}(t)) - i\sum_{i=1}^{d} \mu_i u_i - i \sum_{i \leqslant j=1}^{d} \eta_{ij} p_{ij}) \tag{14}$$

where $u_i = h_{x_i}$ and $p_{ij} = h_{x_i x_j}$. Also introduce q_{ijk} as $q_{ijk} = h_{x_i x_j x_k}$ which will be used later. By considering the zero-tension KPZ equation we will have the following time evolution for h and its derivatives;

$$h_t = \frac{\alpha}{2}\sum_{i=1}^{3} u_i^2 + f \tag{15}$$

$$u_{i,t} = \alpha \sum_{l=1}^{d} u_l p_{li} + f_{x_i} \tag{16}$$

$$p_{ij,t} = \alpha \sum_{l=1}^{d} p_{li} p_{lj} + \alpha \sum_{l=1}^{d} u_l q_{lij} + f_{x_i x_j}. \tag{17}$$

Also using the Novikov's theorem we can write the following identities;

$$\langle f\Theta\rangle = -i\lambda k(0,)Z - i\sum_{l=1}^{3} \eta_{ll} k''(0) Z \tag{18}$$

$$\langle f_{x_i}\Theta\rangle = -i\mu_i k''(0) Z \tag{19}$$

$$\langle f_{x_i x_i}\Theta\rangle = -i\lambda k''(0) Z - i\sum_{l=1}^{3} \eta_{ll} k''''(0) Z \tag{20}$$

$$\langle f_{x_i x_j}\Theta\rangle = -i\eta_{ij} k''''(0) Z \qquad i \neq j \tag{21}$$

where

$$k(x-x', y-y', z-z') = 2D_0 D(x-x', y-y', z-z'),$$
$$k(0)=k(0,0,0) = \frac{2D_0}{(\pi\sigma^2)^{\frac{3}{2}}},$$
$$k_{xxxx}(0,0,0) = \frac{-24D_0}{(\pi\sigma^2)^{\frac{3}{2}}}$$
$$k'(0)=k_x(0,0,0) = k_y(0,0,0)=k_z(0,0,0)=0.$$

Differentiate the generating function Z with respect to t and using the Eqs. (9)–(15) and the following identity

$$iZ_{x_l} - i\lambda Z_{\mu_l} - i\sum_i \mu_i Z_{\eta_{il}} \equiv \sum_{i\leqslant j} \eta_{ij}\langle q_{ijl}\Theta\rangle, \tag{22}$$

time evolution of Z can be written simply as

$$\begin{aligned} Z_t &= i\lambda\gamma(t)Z - i\frac{\lambda\alpha}{2}\sum_l Z_{\mu_l\mu_l} - i\alpha\sum_l Z_{\eta_{ll}} \\ &\quad + i\alpha\sum_{l,i\leqslant j}\eta_{ij}Z_{\eta_{li}\eta_{li}} - \lambda^2 k(0)Z + \sum_l \mu_l^2 k''(0)Z \\ &\quad -2\lambda\sum_l \eta_{ll}k''(0)Z - \left(\sum_{l,k}\eta_{ll}\eta_{kk} + \sum_{l<k}\eta_{lk}^2\right)k''''(0)Z, \end{aligned} \tag{23}$$

where $\gamma(t)$ is defined as $\gamma(t)=\overline{h}_t$. Fourier transforming Z respect to λ, μ_i and η_{ij}, the joint probability density function (PDF) of $\tilde{h}, u_i, p_{ij}$, $P(\tilde{h}, u_i, p_{ij}, t)$ is generated as;

$$\begin{aligned} P(\tilde{h}, u_i, p_{ij}, t) &= \int \frac{d\lambda}{2\pi}\prod_i \frac{d\mu_i}{2\pi}\prod_{i\leqslant j}\frac{d\eta_{ij}}{2\pi}\exp(i\lambda(h(x,y,z,t)+\bar{h}(t)) \\ &\quad + i\sum_l \mu_l u_l + i\sum_{l\leqslant k}\eta_{lk}p_{lk})Z(\lambda, \mu_i, \eta_{ij}, x_i, t) \end{aligned} \tag{24}$$

By the use of Eqs. (23) and (24) the equation governing the $P(\tilde{h}, u_i, p_{ij}, t)$'s time evolution will be derived as

$$\begin{aligned} P_t &= \gamma(t)P_{\tilde{h}} + \frac{\alpha}{2}\sum_l u_l^2 P_{\tilde{h}} - \alpha(d+2)\sum_l p_{ll}P \\ &\quad -\alpha\sum_{l,k\leqslant m} p_{lk}p_{lm}P_{p_{km}} + k(0)P_{\tilde{h}\tilde{h}} - k''(0)\sum_l P_{u_l u_l} \\ &\quad + 2k''(0)\sum_l P_{\tilde{h}p_{ll}} + k''''(0)\sum_{l\leqslant k} P_{p_{lk}p_{lk}} - 2k''''(0)\sum_{l<k} P_{p_{ll}p_{kk}} \end{aligned} \tag{25}$$

where d is the spatial dimension.

Equation (25) enables one to write the time evolution of the moments of height and it's corresponding derivatives. The obtained equation has the following form

$$\begin{aligned}
\frac{d}{dt}\langle\tilde{h}^{n_0}AB\rangle = & -n_0\gamma(t)\langle\tilde{h}^{n_0-1}AB\rangle - \frac{\alpha n_0}{2}\sum_l\langle\tilde{h}^{n_0-1}ABu_l^2\rangle \\
& + \alpha\sum_{l,k\leqslant m} n_{km}\left\langle\tilde{h}^{n_0}AB\frac{p_{lk}p_{lm}}{p_{km}}\right\rangle \\
& - \alpha\sum_l\langle\tilde{h}^{n_0}ABp_{ll}\rangle + k(0)n_0(n_0-1)\langle\tilde{h}^{n_0-2}AB\rangle \\
& - k''(0)\sum_l n_l(n_l-1)\left\langle\frac{\tilde{h}^{n_0}AB}{u_l^2}\right\rangle \\
& + 2k''(0)\sum_l n_0 n_{ll}\left\langle\frac{\tilde{h}^{n_0}AB}{p_{ll}^2}\right\rangle \\
& + k''''(0)\sum_{l\leqslant k} n_{lk}(n_{lk}-1)\left\langle\frac{\tilde{h}^{n_0}AB}{p_{lk}^2}\right\rangle \\
& + 2k''''(0)\sum_{l<k} n_{ll}n_{kk}\left\langle\frac{\tilde{h}^{n_0}AB}{p_{ll}p_{kk}}\right\rangle
\end{aligned} \tag{26}$$

where

$$A = \prod_{i=1}^{d} u_i^{n_i}$$
$$B = \prod_{i\leqslant j} p_{ij}^{n_{ij}}.$$

By choosing different sort of n_0, n_i and n_{ij} values, various type of coupled differential equations, governing the evolution of the moments, can be constructed.

APPENDIX B

In appendix A we have derived the general time evolution equation for the moments for the arbitrary $(d+1)$-dimensional case. The moments in $1+1$ and $2+1$ dimensions has been derived in refs. 8 and 11, respectively. Here we are going to restrict ourselves to the $(3+1)$-dimensional case. Also the moments in $d+1$ dimensions is given at the end of this appendix.

In this appendix the height moments will be calculated exactly and it will be shown that $\langle h_{x_i x_j}\Theta\rangle, i \neq j$ will be zero by considering flat initial condition in the $(3+1)$-dimensional case. However it can be shown that in any general $(d+1)$-dimensional case this identity will also be true. By choosing different sort of n_0, n_i and n_{ij} values, various type of coupled differential equations, governing the evolution of the moments, can be constructed. The first example is to choose $n_0 = n_i = n_{ij} = 0$ then;

$$-\alpha \sum_l \langle p_{ll}\rangle = -\alpha\langle \nabla.\mathbf{u}\rangle = \mathbf{0} \Rightarrow \langle \nabla.\mathbf{u}\rangle = \mathbf{0} \tag{27}$$

which verifies that the fluid is incompressible in average. The main aim is to calculate the moments $\langle p_{ij}\exp(-i\lambda\tilde{h} - i\sum_l \mu_l u_l)\rangle$. Before that, we have to follow several steps. First of all we have to calculate the moments such as $\langle p_{ij}u_iu_j\rangle$ and $i \neq j$. Using the Eq. (20), we have

$$\begin{aligned}\frac{d}{dt}\langle u_iu_jp_{ij}\rangle &= \alpha\sum_l \langle u_iu_jp_{li}p_{lj}\rangle - \alpha\sum_l \langle u_iu_jp_{ll}p_{ij}\rangle \\ &= \alpha\sum_l \langle u_iu_j(p_{li}p_{lj} - p_{ll}p_{ij})\rangle.\end{aligned} \tag{28}$$

Looking at the right hand of Eq. (28) we see that the terms with $l = i$ or $l = j$ cancel each other. Noting that we have restricted ourselves to the $3+1$ dimension, the Eq. (28) can be written

$$\frac{d}{dt}\langle u_iu_jp_{ij}\rangle = \langle u_iu_j(p_{li}p_{lj} - p_{ll}p_{ij})\rangle \qquad i \neq j \neq l \tag{29}$$

Now inserting the RHS of Eq. (29) in Eq. (26), one finds

$$\begin{aligned}&\frac{d}{dt}\langle u_iu_j(p_{li}p_{lj} - p_{ll}p_{ij})\rangle \\ &\quad = \alpha\sum_k \langle u_iu_jp_{ki}p_{kl}p_{lj}\rangle + \alpha\sum_k \langle u_iu_jp_{kj}p_{kl}p_{il}\rangle \\ &\quad\quad -\alpha\sum_k \langle u_iu_jp_{kk}p_{li}p_{lj}\rangle - \alpha\sum_k \langle u_iu_jp_{kl}p_{kl}p_{ij}\rangle \\ &\quad\quad -\alpha\sum_k \langle u_iu_jp_{ki}p_{kj}p_{ll}\rangle + \alpha\sum_k \langle u_iu_jp_{kk}p_{ij}p_{ll}\rangle\end{aligned} \tag{30}$$

It can be easily seen that in the case that $i \neq j \neq l$ the rhs of Eq. (30) will be zero. Using this result and considering a flat initial condition we have;

$$\langle u_iu_j(p_{li}p_{lj} - p_{ll}p_{ij})\rangle = 0 \tag{31}$$

and

$$\langle u_i u_j p_{ij} \rangle = 0. \tag{32}$$

Also it can be shown by induction that all the moments such as $\langle \tilde{h}^{n_0} p_{ij}^{n_{ij}} u_i^{n_i} u_j^{n_j} \rangle$ are zero too. Therefore one concludes that $\langle p_{ij} \exp(-i\lambda\tilde{h} - i \sum_l \mu_l u_l) \rangle = 0$. As shown in appendix C this relation is crucial to derive the Eq. (6).

Now let us calculate the moments of u_i's. Using Eq. (2) it can be shown that

$$\frac{d}{dt} \langle u_i^n \rangle = -\alpha \langle u_i^n \sum_l p_{ll} \rangle - n(n-1)k''(0) \langle u_i^{n-2} \rangle. \tag{33}$$

Differentiating $\langle u_i^{n+1} \rangle$ and $\langle u_i^n u_j \rangle$ with respect to x_i and x_j and using the statistical homogeneity and Eq. (26) we have

$$\langle u_i^n p_{ii} \rangle = \langle u_i^n p_{ii} \rangle = 0. \tag{34}$$

Therefore;

$$\frac{d}{dt} \langle u_i^{n_i} \rangle = -n_i(n_i - 1)k''(0) \langle u_i^{n_i-2} \rangle \tag{35}$$

which is an iterative equation implying that any order of the u_i moment can be calculated by knowing the lower moments. Because $\langle u_i \rangle = 0$, from Eq. (35) it is obvious that any odd moment of u_i will be zero. For $n_i = 2, 4, 6, 8$, $\langle u_i^{n_i} \rangle$ will be

$$\langle u_i^2 \rangle = -2k''(0)t \tag{36}$$

$$\langle u_i^4 \rangle = 12k''(0)^2 t^2 \tag{37}$$

$$\langle u_i^6 \rangle = -120k''(0)^3 t^3 \tag{38}$$

$$\langle u_i^8 \rangle = 1680k''(0)^4 t^4. \tag{39}$$

and $\gamma(t)$ $(\gamma(t) = \overline{h}(t))$ will be;

$$\gamma(t) = \frac{\alpha}{2} \sum_l \langle u_l^2 \rangle = -3\alpha k''(0)t \tag{40}$$

For moments such as $\langle u_i^{n_i} u_j^{n_j} \rangle$, it will be deduced that

$$\frac{d}{dt} \langle u_i^{n_i} u_j^{n_j} \rangle = -n_i(n_i - 1)k''(0) \langle u_i^{n_i-2} u_j^{n_j} \rangle - n_j(n_j - 1)k''(0) \langle u_i^{n_i} u_j^{n_j-2} \rangle - \alpha \left\langle u_i^{n_i} u_j^{n_j} \sum_l p_{ll} \right\rangle \tag{41}$$

Differentiate $\langle u_i^{n_i} u_j^{n_j}\rangle$ and $\langle u_i^{n_i} u_j^{n_j} u_k\rangle$ with respect to x_i and x_k, respectively one finds

$$\langle u_i^{n_i} u_j^{n_j}\rangle_{x_i} = n_i\langle u_i^{n_i-1} u_j^{n_j} p_{ii}\rangle + n_j\langle u_i^{n_i} u_j^{n_j-1} p_{ij}\rangle \qquad i \neq j \tag{42}$$

or

$$\begin{aligned}\langle u_i^{n_i} u_j^{n_j} u_k\rangle_{x_k} &= n_i\langle u_i^{n_i-1} u_j^{n_j} u_k p_{ik}\rangle + n_j\langle u_i^{n_i} u_j^{n_j-1} u_k p_{jk}\rangle \\ &\quad + \langle u_i^{n_i} u_j^{n_j} p_{kk}\rangle \qquad i \neq j \neq k.\end{aligned} \tag{43}$$

Using the statistical homogeneity and identity such as $\langle u_i u_j p_{ll}\rangle = 0$ $(l \neq i, j)$ it can be seen that Eq. (41) would be

$$\begin{aligned}\frac{d}{dt}\langle u_i^{n_i} u_j^{n_j}\rangle &= -n_i(n_i-1)k''(0)\langle u_i^{n_i-2} u_j^{n_j}\rangle \\ &\quad - n_j(n_j-1)k''(0)\langle u_i^{n_i} u_j^{n_j-2}\rangle\end{aligned} \tag{44}$$

Therefore one finds the following expression for the moments;

$$\langle u_i^2 u_j^2\rangle = 4k''^2(0)t^2 \tag{45}$$

$$\langle u_i^4 u_j^2\rangle = -24k''^3(0)t^3 \tag{46}$$

$$\langle u_i^6 u_j^2\rangle = 240k''^4(0)t^4 \tag{47}$$

and

$$\langle u_1^2 u_2^2 u_3^2\rangle = -8k''^3(0)t^3. \tag{48}$$

Now we want to calculate $\langle \tilde{h}^2\rangle$ and its higher moments. Using Eq. (20) it can be shown that the moment $\langle \tilde{h}^2\rangle$ satisfies the following equation

$$\frac{d}{dt}\langle \tilde{h}^2\rangle = -\alpha\sum_l \langle \tilde{h} u_l^2\rangle - \alpha\sum_l \langle \tilde{h}^2 p_{ll}\rangle + 2k(0) \tag{49}$$

But the moment $\langle \tilde{h}^2 p_{ll}\rangle$ can be written as

$$\langle \tilde{h}^2 p_{ll}\rangle = \langle \tilde{h}^2 u_l\rangle_{x_l} - 2\langle \tilde{h} u_l^2\rangle \tag{50}$$

where $\langle \tilde{h}^2 u_l\rangle_{x_l} = 0$, so

$$\langle \tilde{h}^2 p_{ll}\rangle = -2\langle \tilde{h} u_l^2\rangle \tag{51}$$

Then Eq. (49) can be written as

$$\frac{d}{dt}\langle\tilde{h}^2\rangle = \alpha\sum_l\langle\tilde{h}u_l^2\rangle + 2k(0) \tag{52}$$

Before calculating the $\langle\tilde{h}u_l^2\rangle$ moment, the more general term $\langle\tilde{h}^{n_0}u_i^{n_i}\rangle$, can be studied, so

$$\begin{aligned}\frac{d}{dt}\langle\tilde{h}^{n_0}u_i^{n_i}\rangle &= -n_0\gamma(t)\langle\tilde{h}^{n_0-1}u_i^{n_i}\rangle - \frac{n_0\alpha}{2}\sum_l\langle\tilde{h}^{n_0}u_i^{n_i}u_l^2\rangle \\ &\quad -\alpha\sum_l\langle\tilde{h}^{n_0}u_i^{n_i}p_{ll}\rangle + n_0(n_0-1)k(0)\langle\tilde{h}^{n_0-2}u_i^{n_i}\rangle \\ &\quad -n_i(n_i-1)k''(0)\langle\tilde{h}^{n_0}u_i^{n_i-2}\rangle \end{aligned} \tag{53}$$

where

$$\langle\tilde{h}^{n_0}u_i^{n_i}p_{ii}\rangle = -\frac{n_0}{n_i+1}\langle\tilde{h}^{n_0-1}u_i^{n_i+2}\rangle \tag{54}$$

$$\langle\tilde{h}^{n_0}u_i^{n_i}p_{jj}\rangle = -n_0\langle\tilde{h}^{n_0-1}u_i^{n_i}u_j^2\rangle \quad i\neq j \tag{55}$$

Therefore Eq. (53) can be written in a new form as

$$\begin{aligned}\frac{d}{dt}\langle\tilde{h}^{n_0}u_i^{n_i}\rangle &= -n_0\gamma(t)\langle\tilde{h}^{n_0-1}u_i^{n_i}\rangle + \frac{\alpha n_0}{2}\sum_l\langle\tilde{h}^{n_0-1}u_i^{n_i}u_l^2\rangle \\ &\quad -\frac{\alpha n_0}{2}\sum_l\delta_{il}\langle\tilde{h}^{n_0-1}u_i^{n_i}u_l^2\rangle + n_0(n_-1)k(0)\langle\tilde{h}^{n_0-2}u_i^{n_i}\rangle \\ &\quad -n_i(n_i-1)k''(0)\langle\tilde{h}^{n_0}u_i^{n_i-2}\rangle \end{aligned} \tag{56}$$

From Eq. (56), $\langle\tilde{h}u_l^2\rangle$ can be easily obtained

$$\frac{d}{dt}\sum_l\langle\tilde{h}u_l^2\rangle = -\gamma(t)\sum_l\langle u_l^2\rangle - \frac{\alpha}{6}\sum_l\langle u_l^4\rangle + \sum_{l<k}\langle u_l^2u_k^2\rangle \tag{57}$$

So finally by using Eqs. (36), (37) and (45) one gets;

$$\sum_l\langle\tilde{h}u_l^2\rangle = -4\alpha k''^2(0)t^3 \tag{58}$$

Therefore we find;

$$\langle\tilde{h}^2\rangle = -\alpha^2k''^2(0)t^4 + 2k(0)t \tag{59}$$

Now it can be shown that the moment $\langle\tilde{h}^3\rangle$ satisfies the following equation;

$$\frac{d}{dt}\langle\tilde{h}^3\rangle = -3\gamma(t)\langle\tilde{h}^2\rangle - 3\frac{\alpha}{2}\sum_l\langle\tilde{h}^2u_l^2\rangle - \alpha\sum_l\langle\tilde{h}^3p_{ll}\rangle \tag{60}$$

By spatial differentiation $\langle\tilde{h}^3p_{ll}\rangle$ will be proportional to $\langle\tilde{h}^2u_l^2\rangle$, so

$$\frac{d}{dt}\langle\tilde{h}^3\rangle = -3\gamma(t)\langle\tilde{h}^2\rangle + 3\frac{\alpha}{2}\sum_l\langle\tilde{h}^2u_l^2\rangle \tag{61}$$

and in a similar way the time evolution of moment $\langle\tilde{h}^4\rangle$ can be written as

$$\frac{d}{dt}\langle\tilde{h}^4\rangle = -4\gamma(t)\langle\tilde{h}^3\rangle + 2\alpha\sum_l\langle\tilde{h}^2u_l^2\rangle + 12k(0)\langle\tilde{h}^2\rangle. \tag{62}$$

It is appear that for calculating the moments $\langle\tilde{h}^3\rangle$ and $\langle\tilde{h}^4\rangle$, we should calculate the moments $\langle\tilde{h}^3u_l^2\rangle$ and $\langle\tilde{h}^2u_l^2\rangle$. For $\langle\tilde{h}^3\rangle$ it is necessary to know $\sum_l\langle\tilde{h}^2u_l^2\rangle$, so

$$\begin{aligned}\frac{d}{dt}\sum_l\langle\tilde{h}^2u_l^2\rangle =& -2\gamma(t)\sum_l\langle\tilde{h}u_l^2\rangle - \frac{\alpha}{3}\sum_l\langle\tilde{h}u_l^4\rangle \\ &+2\alpha\sum_{l<k}\langle\tilde{h}u_l^2u_k^2\rangle + 2k(0)\sum_l\langle u_l^2\rangle - 6k''(0)\sum_l\langle\tilde{h}^2\rangle\end{aligned} \tag{63}$$

where $\langle\tilde{h}u_l^4\rangle$ and $\langle\tilde{h}u_l^2u_k^2\rangle$ should be calculated from the following equations

$$\begin{aligned}\frac{d}{dt}\sum_l\langle\tilde{h}u_l^4\rangle =& -\gamma(t)\sum_l\langle u_l^4\rangle - \frac{3\alpha}{10}\sum_l\langle u_l^6\rangle \\ &+\frac{\alpha}{2}\sum_{l\neq k}\langle u_l^4u_k^2\rangle - 12k''(0)\sum_l\langle\tilde{h}u_l^2\rangle\end{aligned} \tag{64}$$

$$\begin{aligned}\frac{d}{dt}\sum_{l<k}\langle\tilde{h}u_l^2u_k^2\rangle =& -\gamma(t)\sum_{l<k}\langle u_l^2u_k^2\rangle - \frac{\alpha}{3}\sum_{l<k}\langle u_l^4u_k^2\rangle \\ &+\frac{3\alpha}{2}\langle u_1^2u_2^2u_3^2\rangle - 4k''(0)\sum_l\langle\tilde{h}u_l^2\rangle.\end{aligned} \tag{65}$$

Using Eq. (52) we find;

$$\langle\tilde{h}^3\rangle = -\frac{8}{5}\alpha^3k''^3(0)t^6. \tag{66}$$

To calculate the moment $\langle\tilde{h}^4\rangle$ one needs the following moments

$$\langle\tilde{h}u_1^2u_2^2u_3^2\rangle = -16\alpha k''^4(0)t^5 \tag{67}$$

$$\sum_l\langle\tilde{h}u_l^6\rangle = -720\alpha k''^4(0)t^5 \tag{68}$$

$$\sum_{l<k}\langle u_l^4u_k^4\rangle = 432k''^4t^4 \tag{69}$$

$$\sum_l\langle u_1^2u_2^2u_3^2u_l^2\rangle = 144k''^4t^4 \tag{70}$$

$$\sum_{l,k\neq l}\langle u_l^6u_k^2\rangle = 1440k''^4t^4 \tag{71}$$

$$\sum_{l,k\neq l}\langle\tilde{h}u_l^4u_k^2\rangle = -288k''^4t^5 \tag{72}$$

$$\sum_l\langle\tilde{h}^2u_l^4\rangle = \frac{364}{5}\alpha^2k''^4(0)t^6 + 72k(0)k''^2(0)t^3 \tag{73}$$

$$\sum_{l\leqslant k}\langle\tilde{h}^2u_l^2u_k^2\rangle = \frac{728}{15}\alpha^2k''^4t^6 + 48k(0)k''^2(0)t^3 \tag{74}$$

$$\sum_l\langle\tilde{h}^3u_l^2\rangle = \frac{212}{35}\alpha^3k''^4t^7 - 24\alpha k(0)k''^2(0)t^4 \tag{75}$$

so, finally by substituting the moments above in Eq. (62) we find;

$$\langle\tilde{h}^4\rangle = -\frac{31}{35}\alpha^4k''^4t^8 - 12\alpha^2k(0)k''^2(0)t^5 + 12k(0)^2t^2 \tag{76}$$

Generalizing this method to the $d+1$-dimensional case by a similar amount of calculations, the second, third and fourth moments can be written as;

$$\langle\tilde{h}^2\rangle = \left(\frac{k^2(0)}{\alpha k''(0)}\right)^{\frac{2}{3}}\left[-\left(\frac{d}{3}\right)\left(\frac{t}{t^*}\right)^4 + 2\frac{t}{t^*}\right]$$

$$\langle\tilde{h}^3\rangle = -\frac{8d}{15}\left(\frac{k^2(0)}{\alpha k''(0)}\right)\left(\frac{t}{t^*}\right)^6$$

$$\langle\tilde{h}^4\rangle = \left(\frac{k^2(0)}{\alpha k''(0)}\right)^{\frac{4}{3}}\left[\left(\frac{1}{3}d^2 - \frac{136}{105}d\right)\left(\frac{t}{t^*}\right)^8 - 4d\left(\frac{t}{t^*}\right)^5 + 12\left(\frac{t}{t^*}\right)^2\right]$$

where

$$t_* = \left(\frac{k(0,0)}{\alpha^2k''^2(0,0)}\right)^{1/3}.$$

APPENDIX C

In this appendix using the identities which have found in appendix B, joint-probability distribution function (PDF) is calculated for zero tension KPZ equation. Similar to appendix B we restrict ourselves to the 3+1 dimensions case.

The zero tension KPZ equation in 3+1 dimensions has the following form;

$$h_t(x,y,z,t) - \frac{\alpha}{2}(h_x^2 + h_y^2 + h_z^2) = f \tag{77}$$

Now defining

$$h_x = u, \quad h_y = v, \quad h_z = w \tag{78}$$

Differentiating the KPZ equation (77) with respect to x,y and z, we have

$$h_t = \frac{\alpha}{2}(h_x^2 + h_y^2 + h_z^2) + f(x,y,t) \tag{79}$$
$$u_t = \alpha(uu_x + vv_x + ww_x) + f_x \tag{80}$$
$$v_t = \alpha(uu_y + vv_y + ww_y) + f_y \tag{81}$$
$$w_t = \alpha(uu_z + vv_z + ww_z) + f_z \tag{82}$$

for h and corresponding velocity fields. The generating function $Z(\lambda, \mu_1, \mu_2, \mu_3, x, y, z, t)$ is defined such as to generate the height and velocity field moments. By introducing Θ as

$$\Theta = exp\big(-i\lambda(h(x,y,z,t) - \bar{h}(t)) - i\mu_1 u - i\mu_2 v - i\mu_3 w\big) \tag{83}$$

The generating function will be written as $Z(\lambda, \mu_1, \mu_2, \mu_3, x, y, z, t) = \langle\Theta\rangle$.

Using the KPZ Eq. (79) and its differentiations (78), (79) and (80) with respect to x, y and z, the time evolution of $Z(\lambda, \mu_1, \mu_2, \mu_3, x, y, z, t)$ can be written as

$$\begin{aligned} Z_t &= i\gamma(t)\lambda Z - i\lambda\frac{\alpha}{2}\langle(u^2+v^2+w^2)\Theta\rangle \\ &\quad -i\alpha\mu_1\langle(uu_x+vv_x+ww_x)\Theta\rangle - i\alpha\mu_2\langle(uu_y+vv_y+ww_y)\Theta\rangle \\ &\quad -i\alpha\mu_3\langle(uu_z+vv_z+ww_z)\Theta\rangle - i\lambda\langle f\Theta\rangle - i\mu_1\langle f_x\Theta\rangle \\ &\quad -i\mu_2\langle f_y\Theta\rangle - i\mu_3\langle f_z\Theta\rangle \end{aligned} \tag{84}$$

where $\gamma(t) = h_t = \frac{\alpha}{2}\langle u^2 + v^2 + w^2\rangle$. By considering statistical homogeneity we have

$$Z_x = \langle(-i\lambda u - i\mu_1 u_x - i\mu_2 v_x - i\mu 3 w_x)\Theta\rangle \tag{85}$$

$$Z_y = \langle(-i\lambda v - i\mu_1 u_y - i\mu_2 v_y - i\mu_3 w_y)\Theta\rangle \tag{86}$$

$$Z_z = \langle(-i\lambda w - i\mu_1 u_z - i\mu_2 v_z - i\mu_3 w_z)\Theta\rangle \tag{87}$$

Because we are in a time regime that the singularities has not been formed yet, the order of partial derivatives can be exchanged $\frac{\partial^2 h}{\partial x_i \partial x_j} = \frac{\partial^2 h}{\partial x_j \partial x_i}$ so $v_x = u_y$, $w_x = u_z$ and $w_y = v_z$. Keeping the definition of Θ (83) in mind, we can easily write

$$i\frac{\partial}{\partial\mu_1}\langle(-i\mu_1 u_x - i\mu_2 v_x - i\mu_3 w_x)\Theta\rangle$$
$$= \langle u_x\Theta\rangle - i\mu_1\langle uu_x\Theta\rangle - i\mu_2\langle uv_x\Theta\rangle - i\mu_3\langle uw_x\Theta\rangle. \tag{88}$$

From Eqs (85) and (88) we have

$$\langle u_x\Theta\rangle - i\mu_1\langle uu_x\Theta\rangle - i\mu_2\langle uv_x\Theta\rangle - i\mu_3\langle uw_x\Theta\rangle$$
$$= -\lambda\frac{\partial}{\partial\mu_1}\langle u\Theta\rangle = -i\lambda Z_{\mu_1\mu_1} \tag{89}$$

And in a similar manner one finds;

$$-i\lambda Z_{\mu_2\mu_2} = \langle v_y\Theta\rangle - i\mu_1\langle vu_y\Theta\rangle - i\mu_2\langle vv_y\Theta\rangle - i\mu_3\langle vw_x\Theta\rangle \tag{90}$$

$$-i\lambda Z_{\mu_3\mu_3} = \langle w_z\Theta\rangle - i\mu_1\langle wu_z\Theta\rangle - i\mu_2\langle wv_z\Theta\rangle - i\mu_3\langle ww_z\Theta\rangle \tag{91}$$

By using the Novikov's theorem the expression $\langle f\Theta\rangle$ and $\langle f_{x_i}\Theta\rangle$ can be written respect to Z (appeared in appendix A). So we have

$$Z_t = i\gamma(t)\lambda Z - i\lambda\frac{\alpha}{2}Z_{\mu_1\mu_1} - i\lambda\frac{\alpha}{2}Z_{\mu_2\mu_2}$$
$$-i\lambda\frac{\alpha}{2}Z_{\mu_3\mu_3} - \alpha\langle u_x\Theta\rangle - \alpha\langle v_y\Theta\rangle - \alpha\langle w_z\Theta\rangle - \lambda^2 k(0,0,0)Z$$
$$+\mu_1^2 k_{xx}(0,0,0)Z + \mu_2^2 k_{xx}(0,0,0)Z + \mu_3^2 k_{xx}(0,0,0)Z. \tag{92}$$

The $\langle u_{x_i}\Theta\rangle$ terms can be written as

$$\langle u_x\Theta\rangle = \frac{i}{\mu_1}\langle\Theta\rangle_x + \frac{i}{\mu_1}\langle(i\lambda u + i\mu_2 v_x + i\mu_3 w_x)\Theta\rangle$$
$$= -i\frac{\lambda}{\mu_1}Z_{\mu_1} - \frac{\mu_2}{\mu_1}\langle v_x\Theta\rangle - \frac{\mu_3}{\mu_1}\langle w_x\Theta\rangle, \tag{93}$$

and similarly for $\langle v_y\Theta\rangle$ and $\langle w_z\Theta\rangle$ we have

$$\langle v_y\Theta\rangle = -i\frac{\lambda}{\mu_2}Z_{\mu_2} - \frac{\mu_1}{\mu_2}\langle u_y\Theta\rangle - \frac{\mu_3}{\mu_2}\langle w_y\Theta\rangle \tag{94}$$

$$\langle w_z\Theta\rangle = -i\frac{\lambda}{\mu_3}Z_{\mu_3} - \frac{\mu_1}{\mu_3}\langle u_z\Theta\rangle - \frac{\mu_2}{\mu_3}\langle v_z\Theta\rangle \tag{95}$$

The terms such as $\langle h_{x_i x_j}\Theta\rangle$ $(i \neq j)$ are the main troublesome terms which appear in the Z's time evolution equation (i.e. $\langle u_y\Theta\rangle$), preventing us to write the Z-equation in a closed form. Fortunately as it is shown in Appendix B, these terms will become zero by considering a flat initial condition.

Therefore Z satisfies the following equation;

$$\begin{aligned} Z_t &= i\gamma(t)\lambda Z - i\lambda\frac{\alpha}{2}Z_{\mu_1\mu_1} - i\lambda\frac{\alpha}{2}Z_{\mu_2\mu_2} - i\lambda\frac{\alpha}{2}Z_{\mu_3\mu_3} \\ &\quad + i\alpha\frac{\lambda}{\mu_1}Z_{\mu_1} - i\alpha\frac{\lambda}{\mu_2}Z_{\mu_2} - i\alpha\frac{\lambda}{\mu_3}Z_{\mu_3} - \lambda^2 k(0,0,0)Z \\ &\quad + \mu_1^2 k_{xx}(0,0,0)Z + \mu_2^2 k_{xx}(0,0,0)Z + \mu_3^2 k_{xx}(0,0,0)Z \end{aligned} \tag{96}$$

In what follows we are going to solve the partial differential equation above, by using a flat initial condition, $h(x,y,z,0) = u(x,y,z,0) = v(x,y,z,0) = w(x,y,z,0) = 0$, which equivalently means to write

$$P(\tilde{h},u,v,0) = \delta(\tilde{h})\delta(u)\delta(v)\delta(w). \tag{97}$$

This means that;

$$Z(0,0,0,t) = 1 \tag{98}$$

A useful and efficient way to solve the Z, time-evolution differential equation is to factorize it in the following manner;[8,11]

$$\begin{aligned} Z(\lambda,\mu_1,\mu_2,\mu_3,t) &= F_1(\lambda,\mu_1,t)F_2(\lambda,\mu_2,t)F_3(\lambda,\mu_3,t) \\ &\quad \times \exp(-\lambda^2 k(0)t) \end{aligned} \tag{99}$$

Then by inserting Eq. (99) in Eq. (96) we obtain

$$\begin{aligned} & F_{1_t}F_2F_3 + F_1F_{2_t}F_3 + F_1F_2F_{3_t} \\ &\quad = i\gamma(t)\lambda F_1F_2F_3 - i\lambda\frac{\alpha}{2}F_2F_{1_{\mu_1\mu_1}}F_2F_3 - i\lambda\frac{\alpha}{2}F_1F_{2_{\mu_2\mu_2}}F_3 \\ &\qquad - i\lambda\frac{\alpha}{2}F_1F_2F_{3_{\mu_3\mu_3}} + i\alpha\frac{\lambda}{\mu_1}F_{1_{\mu_1}}F_2F_3 - i\alpha\frac{\lambda}{\mu_2}F_1F_{2_{\mu_2}}F_3 \\ &\qquad - i\alpha\frac{\lambda}{\mu_3}F_1F_2F_{3_{\mu_3}} - \lambda^2 k(0)F_1F_2F_2 + \mu_1^2 k''(0,0)F_1F_2F_3 \\ &\qquad + \mu_2 k''(0)F_1F_2F_2 + \mu_3 k''(0)F_1F_2F_2. \end{aligned} \tag{100}$$

Therefore;

$$F_t = -i\lambda\frac{\alpha}{2}F_{\mu\mu} + i\alpha\frac{\lambda}{\mu}F_\mu + [\mu^2 k''(0) - i\alpha\lambda k''(0)t]F \tag{101}$$

with the initial condition $F(\lambda, \mu, 0) = 1$. This points out that the height gradients in the three dimensions evolve separately from each other before the shock formations, and they are only coupled with the height field. By Fourier transforming Eq. (101) respect to μ a simpler partial differential equation of order one will appear. This equation can be solved by the method of Characteristics.[8] Finally the solution of F will be

$$F(\mu, \lambda, t) = (1 - \tanh^2(\sqrt{2ik_{xx}(0)\alpha\lambda t}))^{-\frac{1}{4}} \exp\left[-\frac{i}{2}\alpha k''(0)\lambda t^2 - \frac{1}{2} i \mu^2 \sqrt{\frac{2ik_{xx}(0)}{\alpha\lambda}} \tanh(\sqrt{2ik_{xx}(0)\alpha\lambda t}) \right] \tag{102}$$

and

$$\begin{aligned} Z(\lambda, \mu_1, \mu_2, t) &= F(\lambda, \mu_1, t) F(\lambda, \mu_2, t) F(\lambda, \mu_3, t) \exp(-\lambda^2 k(0,0) t) \\ &= (1 - \tanh^2(\sqrt{2ik_{xx}(0)\alpha\lambda t}))^{-\frac{3}{4}} \\ &\quad \times \exp\left[-\frac{1}{2} i (\mu_1^2 + \mu_2^2 + \mu_3^2) \sqrt{\frac{2ik_{xx}(0)}{\alpha\lambda}} \tanh(\sqrt{2ik_{xx}(0)\alpha\lambda t}) \right] \\ &\quad \times \exp\left[-\frac{3i}{2}\alpha k''(0)\lambda t^2 - k(0)\lambda^2 t \right] \end{aligned} \tag{103}$$

By inverse Fourier transformation of the generating function Z, the probability distribution function (PDF) of the height fluctuation can be easily derived;

$$P(\tilde{h}, u, v, w, t) = \int \frac{d\lambda}{2\pi} \frac{d\mu_1}{2\pi} \frac{d\mu_2}{2\pi} \frac{d\mu_3}{2\pi} \exp(i\lambda\tilde{h} + i\mu_1 u + i\mu_2 v + i\mu_3 w) Z \tag{104}$$

Expanding the solution of the generating function in powers of λ, all the $\langle (h - \bar{h})^n \rangle$ moments can be derived. For instance, the first five moments before the sharp valley formations are

$$\langle \tilde{h}^2 \rangle = \alpha^2 k''^2(0) \left(\frac{k(0)}{\alpha^2 k''^2(0)} \right)^{\frac{4}{3}} \left[-\left(\frac{t}{t^*}\right)^4 + 2\frac{t}{t^*} \right] \tag{105}$$

$$\langle \tilde{h}^3 \rangle = -\frac{8}{5} \left(\frac{k^2(0)}{\alpha^2 k''^2(0)} \right) \left(\frac{t}{t^*} \right)^6 \tag{106}$$

$$\langle \tilde{h}^4 \rangle = \alpha^4 k''^4(0) \left(\frac{k(0)}{\alpha^2 k''^2(0)} \right)^{\frac{8}{3}}$$

$$\times\left[-\frac{31}{35}\left(\frac{t}{t^*}\right)^8-12\left(\frac{t}{t^*}\right)^5+12\left(\frac{t}{t^*}\right)^2\right] \tag{107}$$

$$\langle\tilde{h}^5\rangle=-\alpha^5 k''^5(0)\left(\frac{k(0)}{\alpha^2 k''^2(0)}\right)^{\frac{10}{3}} \times\left[\frac{-1072}{315}\left(\frac{t}{t^*}\right)^{10}+32\left(\frac{t}{t^*}\right)^7\right] \tag{108}$$

where

$$t_*=\left(\frac{k(0,0)}{\alpha^2 k''^2(0,0)}\right)^{1/3}.$$

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