Length-Constrained Path-Matchings in Graphs

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The *path-matching* problem is to find a set of vertex- or edge-disjoint paths with length constraints in a given graph with a given set of endpoints. This problem has several applications in broadcasting and multicasting in computer networks. In this paper, we study the algorithmic complexity of different cases of this problem. In each case, we either provide a polynomial-time algorithm or prove that the problem is NP-complete. **© 2002 Wiley Periodicals, Inc.**

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1. INTRODUCTION

Let $G = (V, E)$ be a graph and S be a subset of its vertices. A *path-matching* in G covering S is a collection $\mathscr{C}$ of paths in G, such that every vertex of S is an endpoint of exactly one path in $\mathscr{C}$ and every path in $\mathscr{C}$ has both its endpoints in S. A path-matching is called vertex-disjoint, or edge-disjoint, if the collection of paths has this property. The path-matching problem is to find (if possible) a vertex-disjoint or edge-disjoint path-matching in the given (directed or undirected) graph that covers a given set S. Other restrictions may also be imposed on the path-matching. In this paper, we only consider restrictions on the length of the paths. Other restrictions such as an upper bound on the total length of the paths, the number of vertices used, the maximum degree of a vertex in the subgraph induced by the paths, and the weight of the paths in weighted graphs can also be considered (see Cohen et al. [2] and Wu and Manber [13]).

The path-matching problem, under the name of *pseudo-matching,* was considered by Cohen et al. [2] in the study of broadcasting and multicasting protocols in cut-through routed networks. Cohen et al. [2] provided an algorithm for broadcasting in $\lceil \log n \rceil$ rounds in a cut-through routed network of n nodes, assuming a model called the *line model* [6]. This algorithm is based on finding an edge-disjoint path-matching at each round. The maximum length of the paths in this path-matching is a good measure of the time taken in this round. Therefore, to reduce the total broadcast time, one approach is to find path-matchings with length constraints in each round. Other variants of the line model such as the vertex-disjoint line model were studied by Cohen and Fraigniaud [1].

A graph-theoretic version of the path-matching problem was dealt with in Faudree and Gyárfás [7] and Wu and Manber [13], in which the authors presented some results on finding perfect path-matchings (i.e., path-matchings covering all the vertices) in graphs with constraints on the total or maximum length of the paths. Datta and Sen [5] presented an approximation algorithm for the weighted case of the problem. The relationship between the path-matching problem (with a slightly different definition) and the matroid intersection problem was studied by Cunningham and Geelen [4]. Also, several related problems were studied by Csaba et al. [3] and Yinnone [14]. A generalization of this problem called S-paths was introduced by Mader [10] (see also Schrijver [11]).

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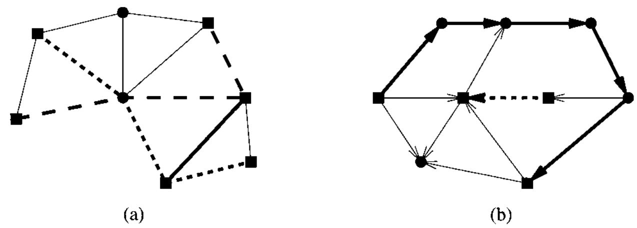

FIG. 1. An example of (a) (U, E, $\ell \leq 3$)-PM and (b) (D, V, $\ell \leq \infty$)-PM. Vertices covered by the path-matchings are marked with squares.

We denote each case of the path-matching problem by a triple. The first component of the triple indicates whether the input graph is directed (D) or undirected (U); the second component indicates whether we want the paths to be vertex-disjoint (V) or edge-disjoint (E); and the third component indicates the restriction on the length of the paths: $\ell \leq k$ means that the length of each path (i.e., the number of edges in the path) is required to be at most k, and $\ell \leq \infty$ means that the path-lengths are unrestricted. Note that, in general, k is a part of the input, unless otherwise stated. For example, (U, E, $\ell \leq 3$)-PM is the problem of finding a path-matching in the given undirected graph such that the paths are edge-disjoint and the length of each path is at most 3 (see Fig. 1).

When we say (D/U, V, $\ell \leq 3$)-PM has a property P (e.g., being NP-complete), it means both problems (D, V, $\ell \leq 3$)-PM and (U, V, $\ell \leq 3$)-PM possess the property P. Cohen et al. [2] proved that

- (U, E, $\ell \leq \infty$)-PM can be solved in polynomial time.
- (U, E, $\ell \leq k$)-PM is NP-complete for general k.

Faudree and Gyárfás [7] proposed a polynomial-time algorithm for (U, E, $\ell \leq 2$)-PM when $S = V(G)$.

In this paper, we prove that (D/U, V/E, $\ell \leq 2$)-PM, (U, V/E, $\ell \leq 3$)-PM, and (D/U, V/E, $\ell \leq \infty$)-PM are polynomially solvable. Also, we prove that (U, V/E, $\ell \leq k$)-PM is NP-complete for any fixed $k \geq 4$ and (D, V/E, $\ell \leq k$)-PM is NP-complete for any fixed $k \geq 3$. Algorithms are presented in Section 2 and NP-completeness results are proved in Section 3.

2. POLYNOMIALLY SOLVABLE SUBPROBLEMS

In this section, we provide efficient algorithms for some of the subproblems of the path-matching problem. In most cases, the algorithm is based on a polynomial-time reduction to the following problem:

TWIN-MATCHING PROBLEM

INSTANCE: An undirected graph G, a subset X of vertices of G, and a set of disjoint pairs $(y_1, y'_1), (y_2, y'_2), \ldots, (y_k, y'_k)$ of vertices of G. The vertices y_i and y'_i are called twins.

Vertices other than $y_1, y'_1, \ldots, y_k, y'_k$ do not have any twin.

QUESTION: Is there a matching M in G such that

- M covers all vertices in X (i.e., every vertex in X is an endpoint of an edge in M), and
- for each pair of twin vertices, M covers either both or neither of them?

First, we prove that the TWIN-MATCHING PROBLEM can be solved in polynomial time.

Lemma 1. *TWIN-MATCHING PROBLEM can be solved in polynomial time.*

Proof. From the given graph G, we construct another undirected graph G' as follows: For each pair of twin vertices (y_i, y'_i), we add two new vertices v_i and v'_i, and edges y_iv_i, $y'_iv'_i$, and $v_iv'_i$ to G. Let G' be the resulting graph. We prove that G has a twin-matching if and only if G' has a matching that covers all the vertices of $X' := X \cup \{y_i, y'_i, v_i, v'_i : i = 1, \ldots, k\}$.

Suppose that G has a twin-matching M. Then, if both of the twin vertices y_i, y'_i are covered by M, we add the edge $v_iv'_i$; otherwise, we add v_iy_i and $v'_iy'_i$ to M. The resulting set of edges is clearly a matching for G' that covers all the vertices of X'. On the other hand, if G' has a matching M' that covers X', it is not difficult to see that $M' \cap E(G)$ is a twin-matching for G.

Now, we construct G'' by adding a clique of size $|V(G')\backslash X'| + b$ to G' and connecting all vertices of this clique to all vertices in $V(G')\backslash X'$, where b is set to either 0 or 1, so that the number of vertices of G'' is even. It is easy to verify that G' has a matching that covers X' if and only if G'' has a perfect matching. Therefore, we can use the $O(|E||V|^{1/2})$ algorithm for finding perfect matchings in general graphs [12] to obtain an $O(|E||V|^{1/2})$ algorithm for the TWIN-MATCHING PROBLEM. ■

Now, we are ready to prove the following theorems:

Theorem 2. *(D, V, $\ell \leq 2$)-PM can be solved in polynomial time.*

Proof. Assume that a directed graph $G = (V, E)$ and $S \subseteq V$ are given. We construct an undirected graph G' from G as follows: Corresponding to each vertex u in S, we put a vertex x_u in G'. Let X be the set of these vertices. Also, for each vertex v in $V\backslash S$, we put two twin vertices y_v, y'_v in G'. Now, for every $u \in S$ and $v \in V\backslash S$, if there is an edge from u to v in G, we connect x_u to y_v in G', and if there is an edge from v to u in G, we connect x_u to y'_v in G'. If for some u, $v \in S$ there is an edge from u to v in G, we put an edge between x_u and x_v in G'.

It is not difficult to see that G has a path-matching with path-lengths at most 2 if and only if G' has a twin-matching. Therefore, Lemma 1 completes the proof. ■

Theorem 3. *$(D, V, \ell \le \infty)$-PM can be solved in polynomial time.*

Proof. We construct an undirected graph G' from the given digraph $G = (V, E)$ and the subset S of V. The construction is similar to the construction in the proof of Theorem 2, with the only difference that here, if for two vertices $u, v \in V \backslash S$ there is an edge from u to v in G, we add the edge $y'_u y_v$ to G'. Notice that there is a one–one correspondence between the edges of G and the edges of G'.

It is easy to see that if there is a path-matching in G covering S the set of corresponding edges in G' will constitute a twin-matching in G'. Conversely, if there is a twin-matching in G', the set of corresponding edges in G constitute a path-matching for G plus a set of disjoint cycles in $V \backslash S$. Therefore, G has a path-matching if and only if G' has a twin-matching. Thus, Lemma 1 completes the proof. ■

Theorem 4. *$(U, V, \ell \le 3)$-PM can be solved in polynomial time.*

Proof. Assume that the undirected graph $G = (V, E)$ and $S \subseteq V$ are given. The graph G' is constructed as follows: Similar to the above proofs, for every $u \in S$, we put a vertex x_u in G', and for every $v \in V \backslash S$, we put two twin vertices y_v and y'_v in G'. Let $X = \{x_v : v \in S\}$. The edges of G' are determined as follows:

- If there is an edge between $u, v \in S$ in G, we put an edge between x_u and x_v in G';
- If there is an edge between $u \in S$ and $v \in V \backslash S$ in G, we connect x_u to both y_v and y'_v in G'; and
- If $u, v \in V \backslash S$ are adjacent in G, we put an edge between y_u and y_v in G'.

It is not difficult to observe that G has a path-matching with path-lengths at most 3, if and only if G' has a twin-matching. Hence, Lemma 1 completes the proof. ■

The following lemma provides a reduction from the undirected to the directed case.

Lemma 5. *For every fixed k, there is a polynomial-time reduction from $(U, V, \ell \le k)$-PM to $(D, V, \ell \le k)$-PM. Also, there is a polynomial-time reduction from $(U, V, \ell \le \infty)$-PM to $(D, V, \ell \le \infty)$-PM.*

Proof. It is sufficient to replace each undirected edge uv with two directed edges, one from u to v and the other from v to u. Clearly, a collection of vertex-disjoint paths in the original graph corresponds to a collection of vertex-disjoint paths in the resulting digraph, with all paths of the same length. ■

Notice that the above reduction does not work in the edge-disjoint case. Using the above lemma and Theorems 2 and 3, we can prove the following corollaries:

Corollary 6. *$(U, V, \ell \le 2)$-PM can be solved in polynomial time.*

Corollary 7. *$(U, V, \ell \le \infty)$-PM can be solved in polynomial time.*

The following lemma provides another reduction among subproblems:

Lemma 8. *For any fixed k, $(U, E, \ell \le k)$-PM can be reduced in polynomial time to $(U, V, \ell \le k + 1)$-PM. Also, $(U, E, \ell \le \infty)$-PM can be reduced in polynomial time to $(U, V, \ell \le \infty)$-PM.*

Proof. For a given graph $G = (V, E)$ and $S \subseteq V$, we construct a graph G' as follows: Corresponding to each edge $e \in E$, we put a vertex y_e in G'. Also, for every vertex $u \in S$, we put a vertex x_u in G'. For every two edges in G that share an endpoint, we connect the corresponding vertices in G' by an edge. If $u \in S$ is an endpoint of $e \in E$, we put an edge between x_u and y_e in G'.

It is easy to see that G has an edge-disjoint path-matching with all paths of length at most k if and only if G' has a vertex-disjoint path-matching with all paths of length at most $k + 1$. ■

It is worth noting that the above lemma can also be generalized for the directed case using essentially the same proof technique. We can use this generalization and Theorems 3 and 7 to prove the following:

Corollary 9. *$(D/U, E, \ell \le \infty)$-PM can be solved in polynomial time.*

We need the following lemma in the proof of Theorem 11:

Lemma 10. *Let $k \le 3$, $G = (V, E)$ be an undirected graph and $S \subseteq V$. If there is a (not necessarily edge-disjoint) path-matching with path-lengths at most k in G covering S, then G also has an edge-disjoint path-matching with path-lengths at most k covering S.*

Proof. Let M be a path-matching with paths of length at most k in G covering S, such that the total length of the paths in M is minimum. We prove that M is edge-disjoint. Assume, for contradiction, that there are two paths P_1 and P_2 in M that are not edge-disjoint. It is easy to see that if we replace P_1 and P_2 with their symmetric difference, we obtain another path-matching M' with a smaller total length of paths. Furthermore, since $k \le 3$, the length of each path in M' is at most k. This contradiction shows that M is edge-disjoint. ■

Now, we can prove the following theorem. Notice that the $k = 2$ case of the following theorem is also a corollary of Theorem 4 and Lemma 8:

Theorem 11. *$(U, E, \ell \leq k)$-PM can be solved in polynomial time for $k \leq 3$.*

Proof. We construct another graph G' as follows: The vertex set of G' is S, and two vertices are adjacent if and only if there is a path of length at most k between them in G. It is clear that G' has a perfect matching if and only if there is a path-matching in G with all path-lengths at most k that covers S. By Lemma 10, this is equivalent to the existence of an edge-disjoint path-matching in G with path-lengths at most k. ■

Theorem 12. *$(D, E, \ell \leq 2)$-PM can be solved in polynomial time.*

Proof. It is easy to see that any path-matching with path-lengths at most 2 is edge-disjoint. Therefore, we can use a technique similar to the proof of Theorem 11. ■

We will prove in the next section that $(D, E, \ell \leq 3)$-PM is NP-complete. However, the next theorem shows that this problem can be solved in polynomial time if S is an independent set.

Theorem 13. *$(D, E, \ell \leq 3)$-PM can be solved in polynomial time for instances (G, S) such that S is an independent set of G.*

Proof. We construct an undirected graph G' from G as follows: Corresponding to each vertex u in S, we put three vertices a_u, b_u, and c_u in G', and for each edge e in G, we put two twin vertices y_e and z_e in G'. For every $u \in S$, we connect c_u to both a_u and b_u. Also, if for some vertex u in S and edge $e = vw$ there is an edge from u to v in G, we put an edge between a_u and y_e, and if $w = u$ or there is an edge from w to u in G, then we put an edge between z_e and b_u. There is no other edge or vertex in G'. Let $X := \{a_u, b_u, c_u : u \in S\}$. We claim that G has a path-matching covering S with path-lengths at most 3, if and only if G' has a twin-matching covering X.

Assume there is a path-matching M in G covering S. For every path $u_1u_2u_3u_4$ of length 3 in M ($u_1, u_4 \in S$), consider the edges $a_{u_1}y_{u_2u_3}$ and $z_{u_2u_3}b_{u_4}$, and for every path $u_1u_2u_3$ of length 2 in M ($u_1, u_3 \in S$), consider the edges $a_{u_1}y_{u_2u_3}$ and $z_{u_2u_3}b_{u_3}$ in G'. Let A be the set of all these edges. It is easy to see that A is a twin-matching, that is, for every pair of twin vertices, it covers either both or none of them. Furthermore, for every $u \in S$, A covers exactly one of the vertices a_u and b_u. Thus, we can match c_u with the unmatched one. By adding all these edges to A, we obtain a twin-matching of G' that covers X.

Conversely, assume that G' has a twin-matching M'. Consider an edge $e = u_2u_3$ in G, for which both y_e and z_e are matched in M'. y_e must be matched with x_{u_1}, for some $u_1 \in S$ that has an edge to u_2. Also, z_e must be matched with either x_{u_3} or x_{u_4} for some $u_4 \in S$ that has an edge from u_3. In the first case, consider the path $u_1u_2u_3$, and in the second case, consider the path $u_1u_2u_3u_4$. Let M denote the set of all these paths. For every $u \in S$, c_u must be matched with exactly one of a_u and b_u. Therefore, u is either the head or the tail of one of the paths in M. Furthermore, for every two paths in M, their second edge is different and therefore they are edge-disjoint. Thus, M constitutes an edge-disjoint path-matching for M that covers S. ■

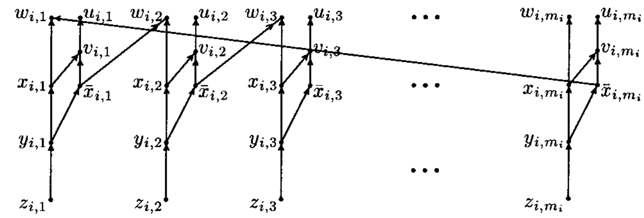

FIG. 2. The component corresponding to x_i in the reduction from EO-3-SAT to $(D, V, \ell \leq 3)$-PM.

3. NP-COMPLETENESS RESULTS

In this section, we prove NP-completeness of several subproblems of the path-matching problem. We use a reduction from the 3-SAT problem (see Garey and Johnson [8]), similar to the reduction that is used by Itai et al. [9]. In fact, we use a reduction from the following restricted version of the 3-SAT problem, in which, for every variable x_i, the number of occurrences of x_i in the formula is equal to that of $\bar{x}_i$. We call this problem Equal Occurrence 3-SAT or EO-3-SAT. It is easy to see that 3-SAT is reducible to this problem (see also Itai et al. [9]).

Theorem 14. *$(D, V, l \leq k)$-PM is NP-complete for any fixed $k \geq 3$.*

Proof. It is clear that the problem is in NP. We prove its NP-completeness by showing a reduction from EO-3-SAT. First, we show this reduction for $k = 3$. Let φ be an EO-3-SAT formula in which the literal x_i (and, therefore, $\bar{x}_i$) has occurred m_i times. We construct a directed graph G and a subset S of its vertices such that φ is satisfiable if and only if G has a vertex-disjoint path-matching covering S with path-lengths at most 3.

Corresponding to each variable x_i, G has one component which has $\{z_{i,j}, y_{i,j}, x_{i,j}, \bar{x}_{i,j}, w_{i,j}, v_{i,j}, u_{i,j} : j = 1, \ldots, m_i\}$ as its vertex set and $\{z_{i,j}y_{i,j}, y_{i,j}x_{i,j}, y_{i,j}\bar{x}_{i,j}, x_{i,j}w_{i,j}, \bar{x}_{i,j}w_{i,(j \bmod m_i)+1}, x_{i,j}v_{i,j}, \bar{x}_{i,j}v_{i,j}, v_{i,j}u_{i,j} : j = 1, \ldots, m_i\}$ as its edge set. This component is shown in Figure 2.

Also, corresponding to each clause C_i in φ, G contains a vertex c_i, and if the rth occurrence of the literal x_i ($\bar{x}_i$) is in C_j, we put a directed edge from c_j to x_{ir} ($\bar{x}_{ir}$, respectively). Now, let U be the set of all $u_{i,j}$'s, W be the set of all $w_{i,j}$'s, Z be the set of all $z_{i,j}$'s, and C be the set of all c_i's. If the size of $U \cup W \cup Z \cup C$ is odd, we add a new vertex to U. For every two vertices $u, u' \in U$, we put an edge from u to u' in G. There is no other vertex or edge in G. Let $S := U \cup W \cup Z \cup C$. We claim that φ is satisfiable if and only if (G, S) is a yes-instance of $(D, V, \ell \leq 3)$-PM.

Assume that φ is satisfiable, and consider a satisfying truth assignment t. For every variable x_i, if $t(x_i) = \text{true}$ (false), consider the set of paths $z_{i,j}y_{i,j}\bar{x}_{i,j}w_{i,(j \bmod m_i)+1}$ ($z_{i,j}y_{i,j}x_{i,j}w_{i,j}$, respectively) for $j = 1, \ldots, m_i$. Let A denote the set of all these paths. Also, for every clause C_j, consider a literal x_i ($\bar{x}_i$) in C_j that is satisfied by t and consider the path $c_jx_{i,r}v_{i,r}u_{i,r}$ ($c_j\bar{x}_{i,r}v_{i,r}u_{i,r}$, respectively), where r is a number such that C_j is the rth occurrence of x_i ($\bar{x}_i$, respectively). Let B denote the set of all these paths. It is clear that $A \cup B$ is a set of vertex-disjoint paths of length 3 that covers Z, W, C, and a subset U' of U. Thus, by adding a perfect matching in $U \backslash U'$ to $A \cup B$, we obtain a path-matching for (G, S). Therefore, (G, S) is a yes-instance of $(D, V, \ell \leq 3)$-PM.

Now, assume that G has a vertex-disjoint path-matching M covering S. Since there is no path of length at most 3 from Z to $S \backslash W$, therefore M must match every vertex of Z with a vertex of W. It is easy to see that, for every i, there are only two possibilities: either $z_{i,j}$ is matched with $w_{i,j}$ by the path $z_{i,j}y_{i,j}x_{i,j}w_{i,j}$ or it is matched with $w_{i,(j \bmod m_i)+1}$ by the path $z_{i,j}y_{i,j}\bar{x}_{i,j}w_{i,(j \bmod m_i)+1}$. In the former case, let $t(x_i) = \mathit{false}$, and in the latter case, let $t(x_i) = \mathit{true}$. Since $|W| = |Z|$, every vertex of W must be matched with a vertex of Z in M. Furthermore, there is no path between two vertices in C. Therefore, every vertex of C must be matched with a vertex of U. It is easy to see that this implies that t is a satisfying assignment for φ.

For $k \geq 4$, it is sufficient to replace the edges $z_{i,j}y_{i,j}$, $c_jx_{i,r}$, and $c_j\bar{x}_{i,r}$ by paths of length $k - 2$. The above argument implies that the resulting graph is a yes-instance of $(D, V, \ell \leq k)$-PM if and only if φ is satisfiable. ∎

Theorem 15. *$(U, V, l \leq k)$-PM is NP-complete for any fixed $k \geq 4$.*

Proof. The proof is very similar to that of Theorem 14. We use the same reduction, with all edges replaced by undirected edges. The only difficulty is in proving that there is no path of length at most k between Z and $S \backslash W$ or between two vertices of C. This can be done by observing that the length of the shortest path between two vertices of Z is $2(k - 2) + 4 \geq k + 1$, the length of the shortest path between a vertex of Z and a vertex of U is $k + 1$, the length of the shortest path between a vertex of Z and a vertex of C is $2(k - 2) + 1 \geq k + 1$ (since $k \geq 4$), and the length of the shortest path between two vertices of C is $2(k - 2) + 2 \geq k + 1$. The rest of the proof is similar. ∎

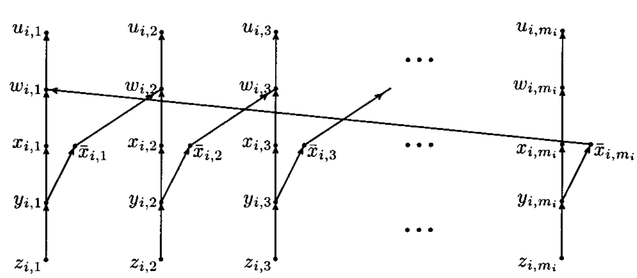

FIG. 3. The component corresponding to x_i in the reduction from EO-3-SAT to $(D, E, \ell \leq 3)$-PM.

TABLE 1. Summary of the results.

k	1	2	3	≥4	∞
$(D, E, \ell \leq k)$-PM	P	P	NPC	NPC	P
$(D, V, \ell \leq k)$-PM	P	P	NPC	NPC	P
$(U, E, \ell \leq k)$-PM	P	P	P	NPC	P
$(U, V, \ell \leq k)$-PM	P	P	P	NPC	P

Theorem 16. *$(D, E, l \leq k)$-PM is NP-complete for any fixed $k \geq 3$ and $(U, E, l \leq k)$-PM is NP-complete for any fixed $k \geq 4$.*

Proof. The proof is very similar to that of Theorems 14 and 15. The only difference is in the component that represents the variable x_i. For the directed case and $k = 3$, this component is shown in Figure 3. For $k > 3$, we replace the edges $z_{i,j}y_{i,j}$, $c_jx_{i,r}$, and $c_j\bar{x}_{i,r}$ by paths of length $k - 2$. For the undirected case, we omit the direction of the edges. In each case, an argument similar to the proof of Theorems 14 and 15 implies that the reduction is correct. ∎

4. CONCLUSIONS

In this paper, we studied several subproblems of the path-matching problem with constraints on the length of the paths. Table 1 shows a summary of our results. There are several open problems related to the path-matching problem. For example, complexity of the path-matching problem for restricted classes of graphs, such as planar graphs, is an open question. Also, similar to Itai et al. [9], one can consider the subproblem in which the length of the paths is restricted to be equal to a given number k. Faudree and Gyárfás [7] defined a *bipartite* version of the path-matching problem in which the set S is partitioned into two subsets S_1 and S_2 and the paths in the path-matching are required to match the vertices of S_1 with the vertices of S_2. All the above problems can also be considered in the bipartite case.

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