Pricing in Population Games with Semi-Rational Agents

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Abstract

We consider a market in which two competing sellers offer two similar products on a social network. In this market, each agent chooses iteratively between the products based on her neighbors reactions and prices. This introduces two games; one between the agents and one between the sellers. We show the first game is a full potential game and provide an algorithm to compute its convergence point. We also study various properties of the second game such as its equilibrium points and convergence.

Keywords: Social Network, Pricing, Population Game, Algorithm, Market

1. Introduction

How can a seller make profit out of a social network? One reasonable policy for monetizing social networks is to spread the product in a population through the network of individual interactions. Because of the rapid growth and popularity of online social networks, the topic has attracted interest among researchers seeking clever policies. For example, several papers have studied agents' behaviors in social markets [1, 2, 3, 8, 10].

In this paper, we study a new model for the market; two competing companies sell two comparable products with networks externalities. Like the classic approach, the social network is modelled by a graph whose edges represent the interaction between people. The main difference, however, is that the nodes of the graph represent communities in the society rather than individuals. Each community consists of a continuum of potential small agents which interact anonymously. So, the market is modeled based on population games [12]. This work studies various related questions such as the behavior of buyers, the strategies of two sellers, price changes, and so on.

In our model, the two companies (sellers) announce their prices first and then, agents within communities choose which company to buy from. An agent's utility depends basically on the fraction of neighbors that are buying the same product as that agent and prices. In our model, agents behave cooperatively in a sense that they tend to buy the same product as most of their friends do. Our aim is to study the behavior of both agents (as consumers) and two competing companies in this game.

To make the setting more realistic, we consider a repeated game in which agents repeatedly revise their decisions. For this, we consider the noisy best-response, logit-response, dynamics for the evolution of the market. In this setting, agents revise their strategies asynchronously. Each agent plays its best-response strategy with some probability close to 1; hence, allowing a slight probability of making mistakes. This may happen in reality when agents' information about the environment are incomplete, when they may make mistakes in their computations, or when agents are not fully rational. The noisy best-response dynamics have been suggested as a method for refining Nash equilibrium in games [6, 9, 4, 5, 10].

Our results. We consider two separate games in our model. The first one is between agents (buyers) who choose between the two products and the second one is between the two companies that announce their prices and sell their products. For the first game, we show that with the logit-response dynamics, the market always converges to an equilibrium point. We show, in Section 3, that the game will be a *full potential game* and its equilibrium point is the global maximum of some potential function. We also prove that agents within the same community buy the same product in the equilibrium. Using this observation, we propose a polynomial-time algorithm for computing the unique equilibrium.

As for the game between the two companies, we study the behavior of the two companies and obtain several results. We show, in Section 4, that the game has either no pure Nash equilibrium or has a unique one. Then, we consider the best-response dynamics between companies and present a polynomial-time algorithm for computing the best response strategy for the companies. We prove if the equilibrium exists, the best-response dynamics converges to it. We also prove the existence of such equilibrium for some graph classes such as preferential attachment graphs and regular graphs. All missing proofs are in Appendix B.

Related work. In traditional game theory, we make strong assumptions about knowledge of individuals and consider them

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fully aware of others. Evolutionary dynamics, on the other hand, are introduced for relaxing these assumptions. Several works (e.g., [6, 9, 4, 5, 10]) have extensively studied these dynamics and pointed out that introducing perturbations to deterministic processes would create distinctive differences in behavior of dynamics. In a seminal work, Kandori et al. [9] investigate evolutionary noisy best-response dynamics and prove that the dynamics converges to an equilibrium in which all agents adopt the same strategy.

Ellison [5] studied the effect of the underlying graph structure on the game; he specifically discussed convergence time for certain graph classes. Following this work, Montanari et al. [10] studied the logit-response dynamics and made a general and precise connection between the convergence time and the structure of the graph. Our model is inspired by these works with one major difference. Unlike the previous models in which each vertex in these models represents a single agent, vertices in our model correspond to communities. This means that we are not dealing with individuals, rather considering the behavior of a large groups each containing several individuals.

The problem of designing a pricing strategy for a company on a social network is extensively studied in literature (See, e.g., [8, 1, 2, 3]). All these works consider a monopolistic situation in which one single company sells its product and tries to maximize its profit by employing a clever strategy. Hartline et al. [8] and Akhlaghpour et al. [2] assume naive behavior for consumers. In fact, they study the market with consumers who act myopically and buy the product as soon as they can afford to buy it. They don't make any reasoning about future reaction of their neighbors and their long-term utility. In order to consider more intelligent agents, Ahamdipour et al. [1] and Bimpikis et al. [3] model the market as a game. In these studies, agents are assumed to be fully rational and do not make mistakes. It seems that the correct model of agents' behavior probably lies somewhere between these two extremes of myopic agents and fully rational agents.

2. Model

In our model, we study a society that consists of several large mutually influencing *communities* . Let n be the number of communities and m^i be the mass of people in the i^{th} community. For a subset T of communities, let $m^T = \sum_{i\in T} m^i$ be the mass of people in T. Let $m = \sum_i m^i$ be the total mass of the society. We normalize the total mass and assume $m = 1$ through the paper. This assumption does not hurt our result. We model the interaction between different communities by an undirected graph $G = (V, E)$ whose nodes correspond to communities and an edge $\{i, j\}$ represents an interaction between communities i and j. We call this graph the *market graph*. We also allow loops, i.e. edge $\{i, i\}$, in G to emphasize that agents in a same community influence each others as well. Let $N(i)$ be the set of neighbors of community i including itself. For two subsets X and Y of communities we define $\delta(X, Y) = \sum_{i\in X}\sum_{j\in Y,(i,j)\in E} m^i m^j$ which represents the amount of interaction between the communities in X and Y. Note that X and Y may have non-empty intersection. We will also use $\delta(X)$ for $\delta(X, X)$ for simplicity.

	A	B
A	a	c
B	d	b

Table 1: The payoff matrix $\mathbf{U}$

Assume there are two products A and B offered by two competing companies with prices p_A and p_B, respectively. Each agent chooses either A or B; so, its strategy space is the set $S = \{A, B\}$. Let x^i_s, where $s \in S$, be the fraction of people in the community i that buy product s. Thus, $x^i_A + x^i_B = m^i$ and $x = (x^i_s)$ is a vector of $2 \times n$ elements representing the strategy profile of the game. We define $m_s(x) = \sum_i x^i_s$ to be the mass of population who use product $s \in S$. Let $\mathcal{D}^i_s(x) = \sum_{j\in N(i)} x^j_s$ be the mass of neighbors of community i that use product s, for $s \in S$. Also, for every $s \in S$, define $\mathcal{D}_s(x) = \sum_{i\in V} x^i_s \mathcal{D}^i_s(x)$. The utility of every person is obtained by aggregating its utility against every single agent that he interacts with. Let $\mathbf{U}$ (illustrated in Table 1) be the payoff matrix for two players. Then, the utility of a person in community i that plays s in a game with strategy profile x would be:

$$F^i_s(x) = \mathbf{U}(s, A)\mathcal{D}^i_A(x) + \mathbf{U}(s, B)\mathcal{D}^i_B(x) - p_s \qquad (1)$$

We assume in our model that $\mathbf{U}$ is symmetric, i.e. $c = d$ and the game defined by matrix $\mathbf{U}$ is a coordination game, i.e. the players obtain a higher payoff by adopting same strategy. In other words, we have $a > d$ and $b > c$. Without loss of generality and throughout the paper, let $a > b$. Also, for the rest of this paper, we assume that $c = d = 0$; we will prove, in Theorem 1, that this assumption does not hurt the generality of our results.

Our game is in category of *population games* which provide a general framework for studying the strategic interactions in which society consists of several populations. The behaviors of agents in each population are the same. In these games, the number of agents is large, impact of each individual agent is small, and agents interact anonymously, i.e., each agent's payoff depends solely on the distribution of opponents' choices. Consider a classic finite game with N players with mass of $\frac{1}{N}$ for each agent, we model a game with continuum of potential small agents by a finite game when $N \to \infty$. For more details on population games see [12].

2.1. Market Dynamics

As mentioned before, two competing companies are offering products A and B with prices p_A and p_B respectively. In a normal situation, agents update their strategies by looking at their neighbors and buy a product that maximizes their benefit. In our model, we consider *noisy best-response dynamics* in which agents adopt their best response with probability close to one. Therefore, there is a slight possibility of making mistakes by agents. More specifically, we study *logit-response dynamics*. For specific treatment of these dynamics in the context of evolutionary game theory, one can refer to [12].

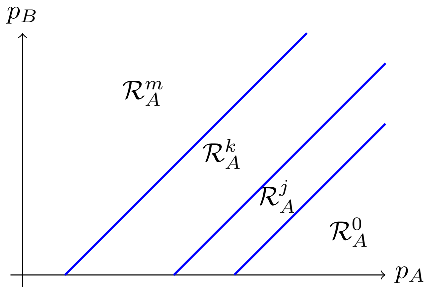

Figure 1: A game with four regions $\mathcal{R}_A^0$, $\mathcal{R}_A^j$, $\mathcal{R}_A^k$, and $\mathcal{R}_A^m$, where $0 < j < k < m$.

In our model, the logit-response dynamics is specified by a parameter $\beta \in R^+$ representing how noisy the system is. In fact, $\beta = \infty$ represents the noise-free or best-response dynamics, and $\beta = 0$ represents the full noisy dynamics in which agents play with no preference. We assume that each agent in a community revises its strategy by arrival of Poisson clock of rate 1. We consider logit-response as revision protocol. So, the probability that an agent in community i takes action s is:

$$P_{i,\beta}(s|x) = \frac{e^{\beta F_s^i(x)}}{\sum_{s' \in S} e^{\beta F_{s'}^i(x)}} \tag{2}$$

As we see later, this game is a *full potential game*, with some potential function f. We have shown that when agents use the logit-response protocol, and when $\beta \to \infty$ the market converges to the global maximum of f. In other words, the dynamics spends most of its time on the global maximum of f. We name this point the *stationary state* of the market.

We now prove that when agents use logit-response as revision protocol, then assuming $c = d = 0$ does not make any difference in our results.

Theorem 1. *Suppose $w \leq \min(a, b, c, d)$. Agents' decisions in game defined on matrix $\mathbf{U}$ is equivalent to agents' decisions in game with matrix $\mathbf{U} - w$, in which $\mathbf{U} - w$ is computed by subtracting w from all the entries of $\mathbf{U}$.*

Given p_A and p_B, we represent the stationary state of the market by $x(p_A, p_B)$ meaning that the game will eventually converge to the strategy profile $x(p_A, p_B)$. We will later see that $x(p_A, p_B)$ depends solely on the difference of p_A and p_B; i. e., if $p_A - p_B = p'_A - p'_B$ then $x(p_A, p_B) = x(p'_A, p'_B)$. We say that profile (p_A, p_B) falls in the region $\mathcal{R}_A^y = \mathcal{R}_B^{m-y}$, if $m_A(x(p_A, p_B)) = y$ and $m_B((p_A, p_B)) = m - y$; i.e., the mass y of the society is using technology A at the stationary state $x(p_A, p_B)$. It is easy to see that increasing p_A decreases y (as depicted in Fig. 1) and since $a > b$, $x(0, 0) \in \mathcal{R}_A^m$.

2.2. Market Pricing Game

Our model introduces a game/competition between the two companies A and B. If $x(p_A, p_B) \in \mathcal{R}_A^y = \mathcal{R}_B^{m-y}$ then the utility (profit) of companies A and B are $U_A(p_A, p_B) = yp_A$ and $U_B(p_A, p_B) = (m - y)p_B$, respectively. The best response for the company A is the price p which maximizes $U_A(p, p_B)$; i. e., $br_A(p_B) = argmax_p U_A(p, p_B)$. Similarly, $br_B(p_A) = argmax_p U_B(p_A, p)$. In the *Market Pricing Game*, we study the game between the two companies and its properties such as its best response behavior and existence of equilibria. We also consider the convergence of the best response dynamics of the game.

3. Market Behavior

In this section, we analyze the behavior of communities when the two companies set prices to p_A and p_B. This will later help us study the market pricing game. First, we show that our game is a *full potential game*, as defined in [11], and has various nice properties. So, the maximizer of potential function will characterize the market stationary state when $\beta \to \infty$. We then use this property to find the market stationary state. We show that the stationary state is very simple when $p_A \leq p_B$. In this case, in the stationary state all agents playing strategy A. But the problem is not trivial when $p_A > p_B$. In this case, we design a polynomial time algorithm that characterizes the stationary state of the market.

3.1. Full Potential Games

Our main result of this section is that our game is a full potential game. We use the following definition from [11]. For more details and useful intuitions, refer to the main article.

Definition 1. *Let $F : \mathbb{R}_+^n \to \mathbb{R}^n$ represent a population game. We call F a* **full potential game** *if there exist a continuously differentiable function $f : \mathbb{R}_+^n \to \mathbb{R}$ satisfying*

$$\nabla f(x) = F(x), \forall x \in \mathbb{R}_+^n \tag{3}$$

In potential games we can capture all information about agents incentives in a scalar valued function, called *potential function*. Existence of such function provides us with many nice properties and enables us to derive various results about our model. In our model, the function F takes a vector x of $2n$ values (x_s^i's) and output the utilities, i.e., the vector of F_s^i's. We prove that our game is full potential by simply finding an f that satisfies equation (3).

Theorem 2. *The function f defined below is the potential function for the game F defined on graph $G = (V, E)$ with payoff matrix $\mathbf{U}$:*

$$f(x) = \frac{1}{2}\left(a\mathcal{D}_A(x) + b\mathcal{D}_B(x)\right) - p_A m_A(x) - p_B m_B(x) \tag{4}$$

3.2. Market Stationary State

In this section, we study the stationary state of the market. First, we provide a lemma that relates the global maximum of potential function to the stationary state of the market. Then we characterize the global maximum of potential function f for the case that $p_A \leq p_B$. Finally, we will study the case $p_A > p_B$ which is more complicated.

As stated before, we consider logit-response dynamics. In this case, our game has a nice property described in the following lemma. In fact, for the case of a continuum of agents in each population this lemma needs an elaborate explanation which is done in Appendix A.

Lemma 3. *In our model, when $\beta \to \infty$, the stationary state of the market is the global maximum of potential function.*

So, in order to estimate the outcome of the game we only need to characterize the global maximum of f. First, we show in Proposition 4 that the stationary state is the state of all agents playing strategy A, when $p_A \leq p_B$.

Proposition 4. *The logit-response dynamics will converge to the state of all agents playing strategy A, if $p_A \leq p_B$ and $\beta \to \infty$.*

Computing the stationary state is more complicated when $p_A > p_B$. In order to solve the problem in this case, we first show in Lemma 5 that in the long run each community will be *homogeneous*, i.e. all people within same community buy same product. This fact helps us to predict the stationary state of the market in polynomial time in Theorem 6. The idea is to build a weighted graph whose minimum cut characterizes the stationary state. The proof is omitted here and appears in Section 5.1.

Lemma 5. *In the logit-response dynamics each community will be homogeneous in the long run, when $\beta \to \infty$.*

Theorem 6. *We can predict the stationary state of the market in polynomial time in the logit-response dynamics, when $\beta \to \infty$.*

4. Market Pricing Game

In this section we consider the game between two competing companies. First, We show that the game has either no pure Nash equilibrium or has a unique one in which $p_B = 0$ in Section 4.1. Then, we consider the best-response dynamics in Section 4.2. We show that each player's best response could be computed in reasonable amount of time. In fact, we introduce a polynomial time algorithm in number of communities in which each company, knowing its opponent's price, can compute the most profitable response. We also prove that if the game has a pure Nash equilibrium then the best-response dynamics will converge to it. At last in Section 4.3, we try to show that in the real world, the market pricing game have a unique Nash equilibrium and the best-response dynamics will converge to it.

It is worth mentioning that, in this setting, we can model monopolistic markets by just setting b and the price of product B to zero. So it is just one company in the market who should decide the best price for its product.

4.1. Pure Nash Equilibrium

In this section, we study equilibrium aspects of our pricing game. Given the results of previous sections, the game between the companies could be simplified as follows. Two companies announce two prices p_A and p_B. The maximum of f is computed. As stated in Lemma 5, every community would be homogeneous in the long run. Let S_A be the set of communities who buy A and $S_B = V - S_A$ be those who buy B. The utilities of the two communities are $p_A m^{S_A}$ and $p_B m^{S_B}$, respectively.

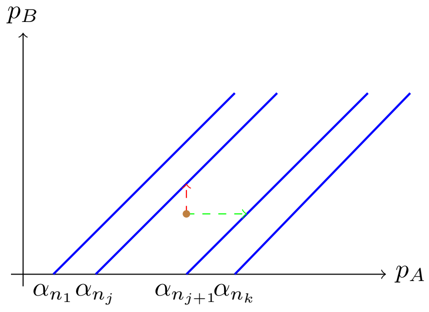

Figure 2: The action of company A (B) has been shown by green(red) line.

In homogeneous state of the market, we can write f as follows.

$$\begin{aligned} f(x) &= \frac{1}{2}(a\delta(S_A) + b\delta(S_B)) - p_A m^{S_A} - p_B m^{S_B} \\ &= f_\delta - f_v - \mathcal{C} \end{aligned}$$

where $f_\delta = \frac{1}{2}(a\delta(S_A) + b\delta(S_B))$, $f_v = (p_A - p_B)m^{S_A}$ and $\mathcal{C} = p_B m$. Since $\mathcal{C}$ is a constant independent of S_A, maximizing f is equivalent to maximizing $f_\delta - f_v$. Note that f_δ is independent of p_A and p_B, and solely depends on the structure of the graph. Let $f_\delta^y = \max_{m^{S_A}=y} f_\delta$. Assume $f_\delta^y = 0$, if there is no set S_A with $m^{S_A} = y$. Therefore, when p_A and p_B is fixed, maximizing f is equivalent to finding y that maximizes $f_\delta^y - y\alpha$, in which $\alpha = p_A - p_B$.

Let (p_A, p_B) be a strategy profile of the pricing game. When $\alpha = 0$ then by Proposition 4 all communities adopt A and, hence, $S_A = m$. As α increases, less communities buy A. Let α_{n_i} be the first point that when $\alpha = \alpha_{n_i}$ then the mass of communities buy A changes to some new value n_i. Let the set of *threshold points* be $\alpha_{n_1} < \alpha_{n_2} < \cdots < \alpha_{n_k}$. For convenience we add $\alpha_{n_0} = 0$. It is clear that $m = n_0 > n_1 > \cdots > n_k = 0$. So, when $p_A - p_B \in [\alpha_{n_j}, \alpha_{n_{j+1}})$ then $m^{S_A} = n_j$ and the utility of company A is $n_j p_A$. See Fig. 2 for illustration. Now, we are ready to prove Theorem 7.

Theorem 7. *The market pricing game has a unique Nash equilibrium if $br_A(0) < \alpha_{n_1}$. Otherwise, it has no Nash equilibrium.*

Proof. Let (p_A, p_B) be a Nash equilibrium and $\alpha = p_A - p_B$. First, we prove if (p_A, p_B) be a Nash equilibrium then, α is less than α_{n_1}.

If $\alpha_{n_j} < \alpha < \alpha_{n_{j+1}}$ for some $j \geq 1$ then B increases his price until $\alpha = \alpha_{n_j}$ (See Fig. 2). This increases B's payoff as it will not affect the communities that buy B. If $\alpha = \alpha_{n_j}$ for some $1 \leq j < n_k$ then A increases his price until it is slightly less than $\alpha_{n_{j+1}}$. This increases A's payoff as it will not affect the communities that buy A. If $\alpha = \alpha_{n_k}$, i.e. no one buys A, then A can decrease his price until at least one community buys A and brings more utility to A. So, we must have $\alpha < \alpha_{n_1}$.

Consider $\alpha < \alpha_{n_1}$. In this situation, company A is interested to increase his price until it is slightly less than α_{n_1}.

Let this value be $\alpha^-_{n_1}$. This increases A's payoff as it will not change the amount of population that buy A. We argue that $p_B = 0$ as if not B can decrease its price to 0 and the new α would be at least α_{n_1} which means some communities buy B and B gets more utility. So, the only possible Nash equilibrium is $(\alpha^-_{n_1}, 0)$. At this point, B is obviously playing best response as he does not get any utility no matter how he plays. However, A necessarily is not playing best response as he may gain more profit by increasing his price. So, we conclude the theorem.

Note that if the strategy domain of companies is continuous then there is not any Nash equilibrium as company A wants to make α as close to α_{n_1} as possible which gives no Nash equilibrium. But, if we discrete the strategy domain then the only possible α is the largest value (in the discrete domain) less than α_{n_1}.

4.2. Best-response Dynamics

In this section, we explore the best-response dynamics of market pricing game. An interesting and important question that we can resolve is computing the best response strategy of companies in the market pricing game. Given the price of company B, p_B, what price p_A should the company A set so as to benefit most? We propose an polynomial time algorithm to answer this question in Theorem 8. The proof of Theorem 8 will appear in Section 5.2.

Then, we study the convergence of best-response dynamics. We show in Theorem 9 that under some conditions the best-response dynamics converges to an equilibrium. Note that the condition of Theorem 7 and 9 are the same which results an interesting property of the game. In fact, we show that if the game has an equilibrium, then it is unique and the best response dynamics will converge to it.

Theorem 8. *In the market pricing game each company, knowing its opponent product price, can determine the best price in polynomial time in number of communities.*

Theorem 9. *The best-response dynamics converge to the unique Nash equilibrium if and only if $br_A(0) < \alpha_{n_1}$.*

Proof. First, we have shown in Theorem 7 that if $br_A(0) \geq \alpha_{n_1}$ the game has no Nash equilibrium. So, we consider the case $br_A(0) < \alpha_{n_1}$.

Let $br_A(0) = \alpha^-_{n_1} < \alpha_{n_1}$. First, we prove $br_A(p_B) = \alpha^-_{n_1} + p_B$. Assume $br_A(p_B) = p \neq \alpha^-_{n_1} + p_B$. Let the profile $(p, p_B) \in \mathcal{R}^y_A$. i.e. the population of agents who buy from company A is exactly y at the profile (p, p_B). Note that points $(\alpha^-_{n_1}, 0)$ and $(\alpha^-_{n_1} + p_B, p_B)$ are in the same region. And $(\alpha^-_{n_1}, 0) \in \mathcal{R}^m_A$. So the $(\alpha^-_{n_1} + p_B, p_B) \in \mathcal{R}^m_A$. Note $br_A(p_A) = p$. Therefore $m(\alpha^-_{n_1} + p_B) < yp$. On the other hand, points (p, p_B) and $(p - p_B, 0)$ are in the same region. So $(p - p_B, 0) \in \mathcal{R}^y_A$. We know $br_A(0) = \alpha^-_{n_1}$. It means that the price $\alpha^-_{n_1}$ is better that the price $p - p_B$ in this case. So $m\alpha^-_{n_1} > y(p - p_B)$, which contracts the fact $m(\mathcal{T}^-_m + p_B) < yp$.

We have shown that $br_A(p_B) = \alpha^-_{n_1} + p_B$. In other words, the best response of company A is to get all the market. So at each state he moves to the right most boundary of the region $\mathcal{R}^m_A$ which the profit of company B is 0. So company B decreases his price and moves out of region $\mathcal{R}^m_A$. They will decrease prices iteratively, until the price of company B becomes 0. Note that $br_A(0) = \alpha^-_{n_1}$, and company A changes the profile to $(\alpha^-_{n_1}, 0)$. We have proved in Theorem 7 that this point is a pure Nash equilibrium point and no one like to change his strategy at the equilibrium point.

4.3. Market equilibrium on special graphs

In this section we show that pure Nash equilibrium exists for some special graphs such as regular and preferential attachment graphs. We first obtain the following sufficient condition for having a Nash equilibrium and then prove it for the above class of graphs. Recall that $f^y_\delta = \max_{m^{S_A}=y} f_\delta$.

Lemma 10. *If $mf^y_\delta < yf^m_\delta + (m - y)f^0_\delta$ for every $y < m$, then $br_A(0) = \alpha^-_1$ and the market has a unique equilibrium*

We conclude this section by showing that several real world market graphs satisfy the condition of Lemma 10 and have pure Nash equilibrium. For the theorem below we consider *uniform markets*, in which we assume the that all populations masses are similar i.e. we have a uniform distribution of agents among populations. In fact, we assume there are n communities in the market with $m_i = 1$. So the total mass of society is $m = n$. This game is important when we want to focus on the structure of the market graph.

Theorem 11. *For the uniform markets, if market graph is a regular or preferential attachment then it has a Nash equilibrium.*

5. Algorithmic Aspects

In this section we propose polynomial time algorithms for two problems. First, we consider the problem of computing the stationary state. The main result is the proof of Theorem 6. Second, we propose a polynomial time algorithm for computing best response for companies in the market pricing game. The main result is the proof of Theorem 8.

5.1. Computing the Stationary State

Let p_A and p_B be fixed. As we know from Lemma 3, the market converges to the maximum of the potential function f. Note that, we have shown in Proposition 4 that in the stationary state, all agents will play strategy A, when $p_A \leq p_B$. So, we focus on the case $p_A > p_B$ and propose a polynomial-time algorithm to compute such a maximum. Our solution is based on an algorithm for the *Maximum Weighted Set Problem*. This problem has been defined below.

Definition 2. Maximum Weighted Set Problem (MWSP): *we are given a directed graph $G = (V, E)$ with (possibly negative) weights I_i on vertices, and non-negative weights w_{ij} on edges. The aim is to find a subset $S \subseteq V$ to maximize $W_S = \sum_{i \in S} I_i + \sum_{\substack{(i,j)\in E \\ i,j \in S}} w_{ij}$.*

Lemma 12. *The MWSP can be solved in polynomial time.*

Proof. The idea is to build a weighted graph whose minimum cut is the solution to the MWSP. For every node i, let $h_i = I_i + \sum_{j\in N(i)} w_{ij}$. We build a graph G' out of G as follows. Add two new nodes s and t. For every i with $h_i < 0$ add an edge with weight $-h_i$ from i to t. For every vertex i with $h_i \geq 0$ add an edge from s to i of weight h_i. The value of the out-cut from any set S which contains s is: $\partial^+(S) = \sum_{\substack{h_i>0\\ i\in T}} h_i + \sum_{\substack{h_i<0\\ i\in S}} -h_i + \sum_{\substack{(i,j)\in E\\ i\in S, j\in T}} w_{ij}$, where $T = V(G') - S$. Let $W = \sum_{h_i>0} h_i = \sum_{\substack{h_i>0\\ i\in S}} h_i + \sum_{\substack{h_i>0\\ i\in T}} h_i$. We rewrite $W - \partial^+(S)$ as:

$$\begin{aligned}
W - \partial^+(S) &= \sum_{\substack{h_i>0\\ i\in S}} h_i + \sum_{\substack{h_i<0\\ i\in S}} h_i - \sum_{\substack{(i,j)\in E\\ i\in S, j\in T}} w_{ij}\\
&= \sum_{i\in S} h_i - \sum_{\substack{(i,j)\in E\\ i\in S, j\in T}} w_{ij}\\
&= \sum_{i\in S}(I_i + \sum_{j\in N(i)} w_{ij}) - \sum_{\substack{(i,j)\in E\\ i\in S, j\in T}} w_{ij}\\
&= \sum_{i\in S} I_i + \sum_{\substack{(i,j)\in E\\ i,j\in S}} w_{ij}
\end{aligned}$$

Since W is a constant independent of S, we conclude that maximizing $\sum_{i\in S} I_i + \sum_{\substack{(i,j)\in E\\ i\in S, j\in T}} w_{ij}$ is equivalent to minimizing $\partial^+(S)$ which could be done in polynomial time.

Lemma 5 helps us to find a connection between MWSP and computing the stationary state. Using this lemma and the algorithm for MWSP we can compute the stationary state of the market and prove Theorem 6.

Proof of Theorem 6: We know from Lemma 5 that each population is homogeneous, so it is suffices to find each population's strategy. We reduce this problem to the MWSP as follows. As proven before, the dynamics of the game converges to the global maximum of the potential function f. Let S_A and S_B be the set of communities in G that play A and B, respectively. We can write potential function (4) for this state of the game as below:

$$f = \frac{1}{2}(a\delta(S_A) + b\delta(S_B)) - p_A m^{S_A} - p_B m^{S_B} \qquad (5)$$

By replacing m^{S_B} by $m - m^{S_A}$ and $\delta(S_B)$ by $\delta(V) - \delta(S_B, S_A) - \delta(S_A, S_B) - \delta(S_A)$ we have:

$$\begin{aligned}
f &= \frac{1}{2}(a\delta(S_A) + b\delta(V) - b\delta(S_B, S_A)\\
&\quad -b\delta(S_A, S_B) - b\delta(S_A))\\
&\quad -p_A m^{S_A} + m^{S_B} - p_B m
\end{aligned}$$

By omitting constant terms that do not affect the maximization, the problem reduces to the problem of finding set S_A to maximize the following statement:

$$\begin{aligned}
&\frac{1}{2}\left((a-b)\delta(S_A) - b\delta(S_A, V - S_A)\right.\\
&\left.-b\delta(V - S_A, S_A)\right) + (p_B - p_A)m^{S_A} \qquad (6)
\end{aligned}$$

We show that the above value is the solution to the MWSP on some graphs G_W that is constructed from G as follows. The vertex set of G_W is that of G. The weight I_i of every vertex i is $(p_B - p_A)m^i - bm^i \sum_{j\in N(i)} m^j$ and w_{ij}, for every edge (i,j) is $\frac{1}{2}(a+b)m^i m^j$. For every set $S \subseteq V$ we have:

$$\begin{aligned}
W_S &= \sum_{i\in S}\left((p_B - p_A)m^i - bm^i \sum_{j\in N(i)} m^j\right)\\
&+ \sum_{\substack{(i,j)\in E\\ i,j\in S}}\left(\frac{1}{2}(a+b)m^i m^j\right)\\
&= (p_B - p_A)m^S - b\delta(S, V) + \frac{1}{2}(a+b)\delta(S)\\
&= \frac{1}{2}((a-b)\delta(S) - b\delta(S, V - S)\\
&- b\delta(V - S, S)) + (p_B - p_A)m^S
\end{aligned}$$

It is clear that finding a maximum weighted set in G_W is equivalent to finding a set S_A that maximizes (6) and, hence, maximizes the potential function f. ∎

5.2. Best-response Pricing

In this section, we propose an algorithm for finding the best-response pricing of companies in the market pricing game. Let us fix p_B. We first obtain lower and upper bounds for the best response of A and then compute it by using binary search. We know from Proposition 4 that if $p_A \leq p_B$ then all populations will play A. So the minimum of p_A is obviously p_B. Also the maximum of p_A is the point where no one play A. The following lemma characterizes this point.

Lemma 13. *Global maximum of potential function f is the state of all agents playing strategy B, if for all $i \in V$ we have $p_A > p_B + \frac{1}{2}(a-b)\sum_{j\in N(i)} m^j$. So the maximum of p_A is at most $p_A^{\max} = p_B + \max_i(\frac{1}{2}(a-b)\sum_{j\in N(i)} m^j)$.*

In order to find the best price for A, all we should do, is to search between maximum and minimum values mentioned above. Algorithm 1 finds the best response of company A. Note that we should search in a continous search space for finding best price. Therefore, we discrete the search space by parameter ϵ and accept ϵ devition. We describe this algorithm in the proof of Theorem 8.

Proof of Theorem 8: we know from Lemma 5 that each community is homogeneous. So there are certain points at which if we decrease p_A a little more, at least one population will change its strategy. We call these points as *threshold points*. For more precise definition of threshold points look at Section 4.1. So, one can fix y, as the total mass of populations who buy A and compute the maximum possible value of p_A for which

Algorithm 1 Algorithm for finding best response of company A to price p_B of company B.

1: $i \leftarrow 1, n_0 \leftarrow m$.
2: **while** $n_i \neq 0$ **do**
3: Let p_A^i be the maximum possible price of company A for which at least mass n_i of people buy A. Find this value by binary search and using Theorem 6.
4: Let $\alpha_{n_i} \leftarrow p_A^i - p_B + \epsilon$.
5: $i \leftarrow i+1$.
6: Let n_i be the mass of people at profile $(p_A^i+\epsilon, p_B)$. Find this value by using Theorem 6.
7: **end while**
8: **return** Price p_A^j which maximize $p_A^j \times n_j$, for $0 \leq j < i$.

at least mass of y people buy A. The latter could be done by a simple binary search algorithm. This gives a profit of at least yp_A. Finally, we find this maximum over all values of y and take the maximum.

Note that for each pricing profile Theorem 6 finds each population's strategy in polynomial time. So, if we accept ϵ deviation, we can find each threshold point in $O(n^3 \log \frac{(p_A^{\max} - p_B)}{\epsilon})$ time. In which $O(n^3)$ is for finding minimum-cut, in order to find strategy of each population, as described in proof of Theorem 6. So, line 3 of algorithm takes at most $O(n^3 \log \frac{(p_A^{\max} - p_B)}{\epsilon})$ time in each iteration. Note that in line 6 of algorithm, we find the mass of people who buy A at the next threshold point.

But, the number of threshold points (the number of while iterations) could be exponential. In fact each community could buy product A or B and the mass of communities are not the same. So, it is possible to have 2^n threshold points. We overcome this problem by using Lemma 14. This lemma states that by increasing p_A no population will change its strategy from B to A. So, by increasing p_A one or more communities changes their strategy from A to B and remain on B till end. Therefore, the number of threshold points at most would be equal to number of communities. ∎

Lemma 14. *Let S_A and S'_A be the set of communities that play A in the stationary state of the market when the price of company B is p_B and the price of company A is p_A and $p'_A > p_A$, respectively. Then $S'_A \subseteq S_A$.*

6. Conclusion

We considered a network of communities through which two sellers compete on selling a similar product. We analyzed both games (between communities and between sellers) and obtained several results regarding the equilibria of the game, the convergence problem in the dynamic market and their efficient computation.

One important research direction is to study convergence rates in the above settings: how long does it take to get to or close to the convergence point? Are there some general graph classes for which there exists rapid convergence?

Another interesting problem is to consider more than two sellers in which prediction of the stationary state of market would become much harder. This seems like a challenging but very interesting question. Also in this work we assumed all the communities to be similar in all aspects, except their masses, but other cases like allowing different behavior of different communities could be considered. Also, a similar problem is to different treatments of sellers. For example, a seller be able to offer different prices to different communities.

We have proved a necessary and sufficient condition of having unique Nash equilibrium in market pricing game and try to show that some markets has this condition. It seems interesting to study real world markets and see whenever they have this condition.

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Appendix A. Stationary State of Infinite Population Game

In this section, we justify the difference between definition of finite and infinite population games. Also, we fill the gap between these two definition by stating an important theorem from [12]. Using this theorem, we prove Lemma 3.

We call finite population game F^N, with total population size N, a *finite full potential* game if there exist a full potential function f^N such that

$$F^N(x) = f^N(x) - f^N(x - \frac{1}{N}e_i) \tag{A.1}$$

The vector e_i is the i^{th} standard vector, and the difference $x - \frac{1}{N}e_i$ means one agent decides not to play strategy i. We have the following important theorem, which relates asymptotic behavior of potential games with agent using exponential update rule, when both total population size and noise level go to infinity.

Theorem 15. *Let $\{F^N\}_{N=N_0}^{\infty}$ be a sequence of finite population potential games with scaled potential functions $\{\frac{1}{N}f^N\}_{N=N_0}^{\infty}$ which converge to the the function f. If agents use the exponential update rule, then the sequence of stationary distributions $\mu^{N,\beta}$ satisfy*

$$\begin{aligned}
\lim_{N\to\infty}\lim_{\beta\to\infty}\max_x \left|\frac{1}{N\beta}\log\mu^{N,\beta}(x) - \Delta f(x)\right| &= 0\\
\lim_{\beta\to\infty}\lim_{N\to\infty}\max_x \left|\frac{1}{N\beta}\log\mu^{N,\beta}(x) - \Delta f(x)\right| &= 0
\end{aligned}$$

In which $\Delta f(x) = f(x) - \max_y f(y) \leq 0$ [12].

Theorem above means that as β and N go to infinity, regardless of their order, the stationary distribution $\mu^{N,\beta}(x)$ decreases to zero with the exponential rate of $\Delta f(x)$, which is the difference of every point x with the global maximum of f. As a result we can deduce that the dynamics spends most of its time around the global maximum of f.

Now, we are ready to prove Lemma 3. For this we need to re-establish definition of our model for finite population case. Assume we have N individual of size $\frac{1}{N}$ in our game. There is no difference in definition of payoff function for finite and infinite case, so we have $F^N(x) = F(x)$ as defined in (1). Also, one can verify that the potential function defined below satisfies the definition in (A.1):

$$\begin{aligned}
f^N(x) &= N\left(\frac{1}{2}\left(a\mathcal{D}_A(x) + b\mathcal{D}_B(x)\right) - p_A m_A(x) - p_B m_B(x)\right) + \frac{1}{2}(am_A(x) + bm_B(x))\\
&= Nf(x) + \frac{1}{2}(am_A(x) + bm_B(x))
\end{aligned}$$

Now, one can easily see that the sequence $\{\frac{1}{N}f^N\}$ converges to $f(x)$. Putting this together with Theorem 15, we can conclude Lemma 3.

Appendix B. Missing Proofs

Proof of Theorem 1: By equation (1), an agent's payoff in community i for strategy s using the payoff matrix $\mathbf{U} - w$ is

$$\begin{aligned}
\hat{F}_s^i(x) &= (\mathbf{U}(s,A) - w)\mathcal{D}_A^i(x) + (\mathbf{U}(s,B) - w)\mathcal{D}_B^i(x) - p_s\\
&= F_s^i(x) - w(\mathcal{D}_A^i(x) + \mathcal{D}_B^i(x))
\end{aligned}$$

It is obvious that both F and $\hat{F}$ result in identical behavior i.e. give the same probability $P_{i,\beta}(s|x)$ in (2). ■

Proof of Theorem 2: We have

$$\begin{aligned}
f(x) &= \frac{1}{2}\left(\sum_{i\in V}\sum_{j\in N(i)} ax_A^i x_A^j + \sum_{i\in V}\sum_{j\in N(i)} bx_B^i x_B^j\right)\\
&- p_A\sum_{i\in V} x_A^i - p_B\sum_{i\in V} x_B^i
\end{aligned}$$

Note that, as mentioned before, $N(i)$ includes i itself. The partial derivative of f with respect to arbitrary x_A^i is

$$\frac{\partial f(x)}{\partial x_A^i} = \frac{1}{2}\left(2\sum_{j\in N(i)} ax_A^j\right) - p_A = a\mathcal{D}_A^i(x) - p_A = F_A^i(x)$$

Comparing with (1) the proof is complete. ■

Proof of Proposition 4: Let y be the state of all agents playing strategy A and x be any other state. By equation (4), we can rewrite $f(y)$ as:

$$f(y) = \frac{1}{2}a\mathcal{D}_A(y) - p_A m_A(y) = \frac{1}{2}a\sum_{i\in V}\sum_{j\in N(i)} m^i m^j - p_A m$$

We bound $f(x)$ as follows:

$$\begin{aligned} f(x) &= \frac{1}{2}\left(a\mathcal{D}_A(x) + b\mathcal{D}_B(x)\right) - p_A m_A(x) - p_B m_B(x) \\ &\leq \frac{1}{2}\left(a\mathcal{D}_A(x) + a\mathcal{D}_B(x)\right) - p_A m_A(x) - p_A m_B(x) \\ &= \frac{1}{2}a\sum_{i\in V}\sum_{j\in N(i)} (x_A^i x_A^j + x_B^i x_B^j) - p_A m \end{aligned}$$

Now by knowing that $x_A^i x_A^j + x_B^i x_B^j \leq (x_A^i + x_B^i)(x_A^j + x_B^j) = m^i m^j$, we can conclude $f(x) \leq f(y)$. So, the maximum of f happens at y and, therefore, the dynamics converges to the state of all agents playing A by Lemma 3. ■

Proof of Lemma 5: Fix community i. As we saw in the proof of Lemma 3, when $\beta \to \infty$, the dynamics converges to the global maximum of f. The part of f that depends on population i (i.e. involves x_A^i and x_B^i) is:

$$\begin{aligned} g(x_A^i) &= \frac{1}{2}\left(a x_A^i x_A^i + b x_B^i x_B^i\right) + x_A^i \sum_{\substack{j\in N(i) \\ i\neq j}} a x_A^j + x_B^i \sum_{\substack{j\in N(i) \\ i\neq j}} b x_B^j \\ &\quad - p_A x_A^i - p_B x_B^i \end{aligned}$$

Since $x_B^i = m_i - x_A^i$, $g(x_A^i)$ will be quadratic in x_A^i and the coefficient of ${x_A^i}^2$ is $C = \frac{1}{2}(a+b) > 0$. Therefore, $g(x_A^i)$ takes its maximum on extreme points, i.e. $x_A^i = 0$ or $x_A^i = m^i$. Since x^i's are independent, the maximum of f happens when for every i, $x_A^i = 0$ or $x_A^i = m^i$. ■

Proof of Lemma 10: We prove that under the above conditions there is only one single threshold point, i.e., as α increases the situation changes from *all playing A* to *all playing B*. Let α be a point at which $|S_A| = y$ in the maximum of f. At this point we have $f_\delta^y - y\alpha \geq f_\delta^0 - 0\times\alpha = f_\delta^0$ and $f_\delta^y - y\alpha \geq f_\delta^m - m\alpha$. So, we have $\frac{f_\delta^m - f_\delta^y}{m-y} \leq \alpha \leq \frac{f_\delta^y - f_\delta^0}{y}$ which means $m f_\delta^y \geq y f_\delta^m + (m-y) f_\delta^0$; this contradicts the lemma condition. Therefore, either all or no communities buy A. Obviously, A's best response at this situation is to play α_1^-. So we have a unique Nash equilibrium by Theorem 7. ■

Proof of Theorem 11: It suffices to prove the condition of Lemma 10.

Regular graphs: Assume we have a regular graph of degree d with $e = nd/2 = md/2$ edges. Note that $f_\delta^m = ae$, $f_\delta^0 = be$ and $f_\delta^y < (ady + bd(m-y))/2$. So $m f_\delta^y < md/2(ay + b(m-y)) = e(ay + b(m-y)) = y f_\delta^m + (m-y) f_\delta^0$

Preferential Attachment Graphs: Assume we have a preferential attachment graph with parameter d with $e = nd = md$ edges. In this model each new node creates exactly d edges to the previous nodes. Note that $f_\delta^m = amd$ and $f_\delta^0 = bmd$. On the other hands, consider an induced sub-graph G' with y vertices. Note G' is connected to the $G - G'$ with at least one edge. So, G' has less than yd edges. Therefore $f_\delta^y < ayd + b(m-y)d$, which implies $m f_\delta^y < y f_\delta^m + (m-y) f_\delta^0$. ■

Proof of Lemma 13: Let y be the state of all agents playing strategy B, and x be an arbitrary state. We have:

$$\begin{aligned} f(x) &= \frac{1}{2}\left(a\mathcal{D}_A(x) + b\mathcal{D}_B(x)\right) - p_A m_A(x) - p_B m_B(x) \\ &\leq \frac{1}{2}\left(a\mathcal{D}_A(x) + b\mathcal{D}_B(x)\right) - p_B m - \frac{1}{2}a\sum_{i\in V}\sum_{i\in N(i)} x_A^i m^j + \frac{1}{2}b\sum_{i\in V}\sum_{i\in N(i)} x_A^i m^j \\ &= \frac{1}{2}a\sum_{i\in V}\sum_{j\in N(i)} (x_A^i x_A^j - x_A^i m^j) + \frac{1}{2}b\sum_{i\in V}\sum_{j\in N(i)} (x_B^i x_B^j + x_A^i m^j) - p_B m \\ &\leq \frac{1}{2}b\sum_{i\in V}\sum_{j\in N(i)} m^i m^j - p_B m = f(y) \end{aligned}$$

■

Proof of Lemma 14: Assume $S'_A \not\subseteq S_A$. Note that the set of communities S_A play A in the the stationary state of the market with prices p_A and p_B. Using proof arguments of Theorem 6, we can conclude that S_A is the maximum weighted set of graph G_W with $I_i = (p_B - p_A)m^i - bm^i \sum_{j \in N(i)} m^j$ and $w_{ij} = \frac{1}{2}(a+b)m^i m^j$. So the weight of set S_A is greater than or equal to the weight of set $S_A \cup S'_A$, which means:

$$\begin{aligned} \sum_{i \in S_A} I_i + \sum_{\substack{(i,j)\in E \\ i,j \in S_A}} w_{ij} &\geq \sum_{i \in S_A \cup S'_A} I_i + \sum_{\substack{(i,j)\in E \\ i,j \in S_A \cup S'_A}} w_{ij} \\ \Rightarrow 0 &\geq \sum_{i \in S'_A - S_A} I_i + \sum_{\substack{(i,j)\in E \\ i \in S'_A - S_A, j \in S_A \cup S'_A}} w_{ij} \end{aligned} \tag{B.1}$$

Similarly, we can show that S'_A is the maximum weighted set of graph $G_{W'}$ with $I'_i = I_i - (p'_A - p_A)m^i$ and $w'_{ij} = w_{ij}$. So the weight of set S'_A is greater than or equal to the weight of set $S_A \cap S'_A$, which means:

$$\begin{aligned} \sum_{i \in S_A \cap S'_A} I'_i + \sum_{\substack{(i,j)\in E \\ i,j \in S_A \cap S'_A}} w'_{ij} &\leq \sum_{i \in S'_A} I'_i + \sum_{\substack{(i,j)\in E \\ i,j \in S'_A}} w'_{ij} \\ \Rightarrow 0 &\leq \sum_{i \in S'_A - S_A} I'_i + \sum_{\substack{(i,j)\in E \\ i \in S'_A - S_A, j \in S'_A}} w'_{ij} \end{aligned} \tag{B.2}$$

Because $p'_A > p_A$, we have $I'_i < I_i$, for every $q \leq i \leq n$. On the other hands, we assumed $S'_A \not\subseteq S_A$, which means $|S'_A - S_A| > 0$. So $\sum_{i \in S'_A - S_A} I'_i < \sum_{i \in S'_A - S_A} I_i$. Now using inequalities (B.1, B.2) and the fact $w'_{ij} = w_{ij}$, we conclude $\sum_{\substack{(i,j)\in E \\ i \in S'_A - S_A, j \in S_A - S'_A}} w_{ij}$ is less than zero. This is a contradiction because we know $w_{ij} \geq 0$, for every $0 \leq i, j \leq n$. ■